FUNCTIONAL ANALYSIS

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Abstract. This is a note on the course MATH 4010: Functional analysis in 2023-24, 1st term.

1. Normed spaces

Definition 1.1. *Let X be a vector space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if it satisfies the following conditions.*

(i) $\|x\| \geq 0$ *for all* $x \in X$.
(ii) $\|x\| = 0$ *if and only if* $x = 0$.
(iii) $\|\alpha x\| = |\alpha|\|x\|$ *for* $x \in X$ *and* $\alpha \in \mathbb{K}$.
(iii) (Triangle inequality) $\|x - y\| \leq \|x - z\| + \|z - y\|$ *for all* $x, y, z \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a normed space.

Example 1.2. The following are important examples of finite dimensional normed spaces.

(i) Let $\ell_\infty^{(n)} = \{(x_1, ..., x_n) : x_i \in \mathbb{K}, i = 1, 2..., n\}$. Put $\|(x_1, ..., x_n)\|_\infty = \max\{|x_i| : i = 1, .., n\}$.
(ii) Let $\ell_p^{(n)} = \{(x_1, ..., x_n) : x_i \in \mathbb{K},\ i = 1, 2..., n\}$. Put $\|(x_1, ..., x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$.

Proposition 1.3. *If X is a normed space, then the addition $(x, y) \in X \times X \mapsto x + y \in X$ and the scalar multiplication $(\alpha, x) \in \mathbb{K} \times X \mapsto \alpha x \in X$ both are continuous maps.*

Notation 1.4. *From now on, $(X, \|\cdot\|)$ always denotes a normed space over a field $\mathbb{K}$.*
For $r > 0$ and $x \in X$, let

(i) $B(x, r) := \{y \in X : \|x - y\| < r\}$ *(called an open ball with the center at x of radius r) and* $B^*(x, r) := \{y \in X : 0 < \|x - y\| < r\}$
(ii) $B(x, r) := \{y \in X : \|x - y\| \leq r\}$ *(called a closed ball with the center at x of radius r).*

Put $B_X := \{x \in X : \|x\| \leq 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$ the closed unit ball and the unit sphere of X respectively.

Definition 1.5. *We say that a sequence (x_n) in X converges to an element $a \in X$ if $\lim \|x_n - a\| = 0$, i.e., for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - a\| < \varepsilon$ for all $n \geq N$.*
In this case, (x_n) is said to be convergent and a is called a limit of the sequence (x_n).

Definition 1.6. *Let A be a subset of X.*

(i) A point $z \in X$ is called a limit point of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < \|z - a\| < \varepsilon$, that is, $B^*(z, \varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.
Furthermore, if A contains the set of all its limit points, then A is said to be closed in X.
(ii) The closure of A, denoted by $\overline{A}$, is defined by

$$\overline{A} := A \cup \{z \in X : z \text{ is a limit point of } A\}.$$

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Remark 1.7. *Using the notations as above, a point* $z \in \overline{A}$ *if and only if* $B(z,r) \cap A \neq \emptyset$ *for all* $r > 0$*. This is equivalent to saying that there is a sequence* (x_n) *in* A *such that* $x_n \to a$*. In fact, this can be shown by considering* $r = \frac{1}{n}$ *for* $n = 1, 2....$

Proposition 1.8. *Using the notations as before, we have the following assertions.*

(i) *A is closed in X if and only if its complement $X \setminus A$ is open in X.*
(ii) *The closure $\overline{A}$ is the smallest closed subset of X containing A. The "smallest" in here means that if F is a closed subset containing A, then $\overline{A} \subseteq F$.*
Consequently, A is closed if and only if $\overline{A} = A$.

Proof. If A is empty, then the assertions (i) and (ii) both are obvious. Now assume that $A \neq \emptyset$.
For part (i), let $C = X \setminus A$ and $b \in C$. Suppose that A is closed in X. If there exists an element $b \in C \setminus int(C)$, then $B(b,r) \nsubseteq C$ for all $r > 0$. This implies that $B(b,r) \cap A \neq \emptyset$ for all $r > 0$ and hence, b is a limit point of A since $b \notin A$. It contradicts to the closeness of A. Thus, $C = int(C)$ and thus, C is open.
For the converse of (i), assume that C is open in X. Assume that A has a limit point z but $z \notin A$. Since $z \notin A$, $z \in C = int(C)$ because C is open. Hence, we can find $r > 0$ such that $B(z,r) \subseteq C$. This gives $B(z,r) \cap A = \emptyset$. This contradicts to the assumption of z being a limit point of A. Thus, A must contain all of its limit points and hence, it is closed.

For part (ii), we first claim that $\overline{A}$ is closed. Let z be a limit point of $\overline{A}$. Let $r > 0$. Then there is $w \in B^*(z,r) \cap \overline{A}$. Choose $0 < r_1 < r$ small enough such that $B(w,r_1) \subseteq B^*(z,r)$. Since w is a limit point of A, we have $\emptyset \neq B^*(w,r_1) \cap A \subseteq B^*(z,r) \cap A$. Hence, z is a limit point of A. Thus, $z \in \overline{A}$ as required. This implies that $\overline{A}$ is closed.
It is clear that $\overline{A}$ is the smallest closed set containing A.
The last assertion follows from the minimality of the closed sets containing A immediately.
The proof is complete. □

A sequence (x_n) in X is called a ***Cauchy sequence*** if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$. We have the following simple observation.

Lemma 1.9. *Every convergent sequence in X is a Cauchy sequence.*

The following notation plays an important role in mathematics.

Definition 1.10. A normed space X is called a *Banach space* if it is a complete normed space, i.e., every Cauchy sequence in X is convergent.

Proposition 1.11. *Let X be a normed space. Then the following assertions are equivalent.*

(i) *X is a Banach space.*
(ii) *If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in X, i.e., $\sum_{n=1}^{\infty} \|x_n\| < \infty$, implies that the series $\sum_{n=1}^{\infty} x_n$ converges in the norm.*

Proof. $(i) \Rightarrow (ii)$ is obvious.
Now suppose that Part (ii) holds. Let (y_n) be a Cauchy sequence in X. It suffices to show that (y_n) has a convergent subsequence. In fact, by the definition of a Cauchy sequence, there is a subsequence (y_{n_k}) such that $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k}$ for all $k = 1, 2....$ By the assumption, the series $\sum_{k=1}^{\infty} (y_{n_{k+1}} - y_{n_k})$ converges in the norm, and hence the sequence (y_{n_k}) is convergent in X. The proof is complete. □

Throughout the note, we write a sequence of numbers as a function $x : \{1, 2, ...\} \to \mathbb{K}$.
The following examples are important classes in the study of functional analysis.

Example 1.12. Put

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K},\ \lim |x(i)| = 0\} \text{ (the } \textit{null sequence space}\text{);}$$
$$\ell_\infty := \{(x(i)) : x(i) \in \mathbb{K},\ \sup_i x(i) < \infty \text{ (the } \textit{bounded sequence space}\text{);}$$

and

$$c_{00} := \{(x(i)) : \text{ there are only finitly many } x(i)\text{'s are non-zero}\} \text{ (the } \textit{finite sequence space).}$$

The sup-norm $\|\cdot\|_\infty$ on ℓ_∞ is defined by $\|x\|_\infty := \sup_i |x(i)|$ for $x \in \ell_\infty$. Then ℓ_∞ is a Banach space.
Now if c_{00} is endowed with the sup-norm defined above, then c_{00} is dense in c_0, i.e., $\overline{c_{00}} = c_0$. Consequently, c_0 is a closed subspace of ℓ_∞. In particular, c_0 is Banach space too.

Proof. We first claim that $\overline{c_{00}} \subseteq c_0$. Let $z \in \ell_\infty$. It suffices to show that if $z \in \overline{c_{00}}$, then $z \in c_0$, i.e., $\lim_{i\to\infty} z(i) = 0$. Let $\varepsilon > 0$. Then there is $x \in B(z, \varepsilon) \cap c_{00}$ and hence, we have $|x(i) - z(i)| < \varepsilon$ for all $i = 1, 2.....$ Since $x \in c_{00}$, there is $i_0 \in \mathbb{N}$ such that $x(i) = 0$ for all $i \geq i_0$. Therefore, we have $|z(i)| = |z(i) - x(i)| < \varepsilon$ for all $i \geq i_0$. Therefore, $z \in c_0$ is as desired.

For the reverse inclusion, let $w \in c_0$. We need to show that $B(w, r) \cap c_{00} \neq \emptyset$ for all $r > 0$. Let $r > 0$. Since $w \in c_0$, there is i_0 such that $|w(i)| < r$ for all $i \geq i_0$. If we let $x(i) = w(i)$ for $1 \leq i < i_0$ and $x(i) = 0$ for $i \geq i_0$, then $x \in c_{00}$ and $\|x - w\|_\infty := \sup_{i=1,2...} |x(i) - w(i)| < r$ is as required. □

Example 1.13. For $1 \leq p < \infty$. Put

$$\ell_p := \{(x(i)) : x(i) \in \mathbb{K},\ \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

In addition, ℓ_p is equipped with the norm $\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{\frac{1}{p}}$ for $x \in \ell_p$. Then ℓ_p becomes a Banach space under the norm $\|\cdot\|_p$.

Example 1.14. Let X be a locally compact Hausdorff space, for example, $\mathbb{K}$. Let $C_0(X)$ be the space of all continuous $\mathbb{K}$-valued functions f on X which are vanish at infinity, i.e., for every $\varepsilon > 0$, there is a compact subset D of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus D$. Now $C_0(X)$ is endowed with the sup-norm, i.e.,

$$\|f\|_\infty = \sup_{x\in X} |f(x)|$$

for every $f \in C_0(X)$. Then $C_0(X)$ is a Banach space. (**Try to prove this fact for the case $X = \mathbb{R}$. Just use the knowledge from MATH 2060 !!!**)

Proposition 1.15. *Let $(X, \|\cdot\|)$ be a normed space. Then there is a normed space $(X_0, \|\cdot\|_0)$, together with a linear map $i : X \to X_0$, satisfies the following conditions.*

(i) X_0 is a Banach space.
(ii) The map i is an isometry, that is, $\|i(x)\|_0 = \|x\|$ for all $x \in X$.
(iii) the image $i(X)$ is dense in X_0, that is, $\overline{i(X)} = X_0$.

Moreover, such pair (X_0, i) is unique up to isometric isomorphism in the following sense.
If $(W, \|\cdot\|_1)$ is a Banach space and an isometry $j : X \to W$ is an isometry such that $\overline{j(X)} = W$, then there is an isometric isomorphism ψ from X_0 onto W such that

$$j = \psi \circ i : X \to X_0 \to W.$$

In this case, the pair (X_0, i) is called the completion of X.

Example 1.16. Proposition 1.15 cannot give an explicit form of the completion of a given normed space. The following examples are basically due to the uniqueness of the completion.

(*i*) If X is a Banach space, then the completion of X is itself.
(*ii*) The completion of the finite sequence space c_{00} is the null sequence space c_0.
(*iii*) The completion of $C_c(\mathbb{R})$ is $C_0(\mathbb{R})$.

2. Finite Dimensional Normed Spaces

Throughout this section, let $(X, \|\cdot\|)$ is a normed space. Put S_X the unit sphere of X, i.e., $S_X = \{x \in X : \|x\| = 1\}$.

Definition 2.1. *Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X are* equivalent*, denoted by $\|\cdot\| \sim \|\cdot\|'$, if there are positive numbers c_1 and c_2 such that $c_1\|\cdot\| \leq \|\cdot\|' \leq c_2\|\cdot\|$ on X.*

Example 2.2. *Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on ℓ^1. We want to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent. In fact, if we put $x_n(i) := (1, 1/2, ..., 1/n, 0, 0, ...)$ for $n, i = 1, 2....$ Then $x_n \in \ell^1$ for all n. Note that (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_\infty$ but it is not a Cauchy sequence with respect to the norm $\|\cdot\|_1$. Hence $\|\cdot\|_1 \nsim \|\cdot\|_\infty$ on ℓ^1.*

Proposition 2.3. *All norms on a finite dimensional vector space are equivalent.*

Proof. Let X be a finite dimensional vector space and let $\{e_1, ..., e_n\}$ be a vector basis of X. For each $x = \sum_{i=1}^n \alpha_i e_i$ for $\alpha_i \in \mathbb{K}$, define $\|x\|_0 = \max_{i=1}^n |\alpha_i|$. Then $\|\cdot\|_0$ is a norm X. The result is obtained by showing that all norms $\|\cdot\|$ on X are equivalent to $\|\cdot\|_0$.
Note that for each $x = \sum_{i=1}^n \alpha_i e_i \in X$, we have $\|x\| \leq (\sum\limits_{1\leq i\leq n} \|e_i\|)\|x\|_0$. It remains to find $c > 0$ such that $c\|\cdot\|_0 \leq \|\cdot\|$. In fact, let $S_X := \{x \in X : \|x\|_0 = 1\}$ be the unit sphere of X with respect to the norm $\|\cdot\|_0$. Note that by using the Weierstrass Theorem on $\mathbb{K}$, we see that S_X is compact with respect to the norm $\|\cdot\|_0$.
Define a real-valued function f on the unit sphere S_X of X by

$$f : x \in S_X \mapsto \|x\|.$$

Note that $f > 0$ and f is continuous with respect to the norm $\|\cdot\|_0$. Hence, there is $c > 0$ such that $f(x) \geq c > 0$ for all $x \in S_X$. This gives $\|x\| \geq c\|x\|_0$ for all $x \in X$ as desired. The proof is complete. □

Corollary 2.4. *We have the following assertions.*

(i) All finite dimensional normed spaces are Banach spaces. Consequently, any finite dimensional subspace of a normed space is closed.
(ii) The closed unit ball of any finite dimensional normed space is compact.

Proof. Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Using the notations as in the proof of Proposition 2.3 above, we see that $\|\cdot\|$ must be equivalent to the norm $\|\cdot\|_0$. X is clearly complete with respect to the norm $\|\cdot\|_0$ and so is complete in the original norm $\|\cdot\|$. The Part (*i*) follows. For Part (*ii*), it is clear that the compactness of the closed unit ball of X is equivalent to saying that any closed and bounded subset is compact. Therefore, Part (*ii*) follows from the simple observation that any closed and bounded subset of X with respect to the norm $\|\cdot\|_0$ is compact. The proof is complete. □

In the remainder of this section, we want to show that the converse of Corollary 2.4(ii) holds. Before this result, we need the following useful result.

Lemma 2.5. Riesz's Lemma: *Let Y be a closed proper subspace of a normed space X. Then for each $\theta \in (0,1)$, there is an element $x_0 \in S_X$ such that $d(x_0, Y) := \inf\{\|x_0 - y\| : y \in Y\} \geq \theta$.*

Proof. Let $u \in X - Y$ and $d := \inf\{\|u - y\| : y \in Y\}$. Note that since Y is closed, $d > 0$ and hence we have $0 < d < \frac{d}{\theta}$ because $0 < \theta < 1$. This implies that there is $y_0 \in Y$ such that $0 < d \leq \|u - y_0\| < \frac{d}{\theta}$. Now put $x_0 := \frac{u - y_0}{\|u - y_0\|} \in S_X$. We are going to show that x_0 is as desired. Indeed, let $y \in Y$. Since $y_0 + \|u - y_0\| y \in Y$, we have

$$\|x_0 - y\| = \frac{1}{\|u - y_0\|}\|u - (y_0 + \|u - y_0\|y)\| \geq d/\|u - y_0\| > \theta.$$

Thus, $d(x_0, Y) \geq \theta$. □

Remark 2.6. The Riesz's lemma does not hold when $\theta = 1$. The following example can be found in the Diestel's interesting book without proof (see [6, Chapter 1 Ex.3(i)]).
Let $X = \{x \in C([0,1], \mathbb{R}) : x(0) = 0\}$ and $Y = \{y \in X : \int_0^1 y(t)dt = 0\}$. Both X and Y are endowed with the sup-norm. Note that Y is a closed proper subspace of X. We are going to show that for any $x \in S_X$, there is $y \in Y$ such that $\|x - y\|_\infty < 1$. Thus, the Riesz's Lemma does not hold as $\theta = 1$ in this case.
In fact, let $x \in S_X$. Since $x(0) = 0$ with $\|x\|_\infty = 1$, we can find $0 < a < 1/4$ such that $|x(t)| \leq 1/4$ for all $t \in [0, a]$.
We fix $0 < \varepsilon < 1/4$ first. Since x is uniform continuous on $[a, 1]$, we can find a partitions $a = t_0 < \cdots < t_n = 1$ on $[a, 1]$ such that $\sup\{|x(t) - x(t')| : t, t' \in [t_{k-1}, t_k]\} < \varepsilon/4$. Now for each (t_{k-1}, t_k), if $\sup\{x(t) : t \in [t_{k-1}, t_k]\} > \varepsilon$, then we set $\phi(t) = \varepsilon$. In addition, if $\inf\{x(t) : t \in [t_{k-1}, t_k]\} < -\varepsilon$, then we set $\phi(t) = -\varepsilon$. From this, one can construct a continuous function ϕ on $[a, 1]$ such that $\|\phi - x|_{[a,1]}\|_\infty < 1$ and $|\phi(x)| < 2\varepsilon$ for all $x \in [a, 1]$. Hence, we have $|\int_a^1 \phi(t)dt| \leq 2\varepsilon(1 - a)$.
As $|x(t)| < 1/4$ on $[0, a]$, so if we choose ε small enough such that $(1 - a)(2\varepsilon) < a/4$, then we can find a continuous function y_1 on $[0, a]$ such that $|y_1(t)| < 1/4$ on $[0, a]$ with $y_1(0) = 0; y_1(a) = x(a)$ and $\int_0^a y_1(t)dt = -\int_a^1 \phi(t)dt$. Now we define $y = y_1$ on $[0, a]$ and $y = \phi$ on $[a, 1]$. Then $\|y - x\|_\infty < 1$ and $y \in Y$ is as desired.

Theorem 2.7. *X is a finite dimensional normed space if and only if the closed unit ball B_X of X is compact.*

Proof. The necessary condition has been shown by Proposition 2.4(ii).
Now assume that X is of infinite dimension. Fix an element $x_1 \in S_X$. Let $Y_1 = \mathbb{K}x_1$. Then Y_1 is a proper closed subspace of X. The Riesz's lemma gives an element $x_2 \in S_X$ such that $\|x_1 - x_2\| \geq 1/2$. Now consider $Y_2 = span\{x_1, x_2\}$. Then Y_2 is a proper closed subspace of X since $\dim X = \infty$. To apply the Riesz's Lemma again, there is $x_3 \in S_X$ such that $\|x_3 - x_k\| \geq 1/2$ for $k = 1, 2$. To repeat the same step, there is a sequence $(x_n) \in S_X$ such that $\|x_m - x_n\| \geq 1/2$ for all $n \neq m$. Thus, (x_n) is a bounded sequence without any convergence subsequence. Hence, B_X is not compact. The proof is complete. □

Recall that a metric space Z is said to be *locally compact* if for any point $z \in Z$, there is a compact neighborhood of z. Theorem 2.7 implies the following corollary immediately.

Corollary 2.8. *Let X be a normed space. Then X is locally compact if and only if $\dim X < \infty$.*

3. Bounded Linear Operators

Proposition 3.1. *Let T be a linear operator from a normed space X into a normed space Y. Then the following statements are equivalent.*

(i) T *is continuous on* X.
(ii) T *is continuous at* $0 \in X$.
(iii) $\sup\{\|Tx\| : x \in B_X\} < \infty$.

In this case, let $\|T\| = \sup\{\|Tx\| : x \in B_X\}$ *and* T *is said to be bounded.*

Proof. $(i) \Rightarrow (ii)$ is obvious.
For $(ii) \Rightarrow (i)$, suppose that T is continuous at 0. Let $x_0 \in X$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $\|Tw\| < \varepsilon$ for all $w \in X$ with $\|w\| < \delta$. Therefore, we have $\|Tx - Tx_0\| = \|T(x - x_0)\| < \varepsilon$ for any $x \in X$ with $\|x - x_0\| < \delta$. Part (i) follows.
For $(ii) \Rightarrow (iii)$, since T is continuous at 0, there is $\delta > 0$ such that $\|Tx\| < 1$ for any $x \in X$ with $\|x\| < \delta$. Now for any $x \in B_X$ with $x \neq 0$, we have $\|\frac{\delta}{2}x\| < \delta$. Therefore, we see have $\|T(\frac{\delta}{2}x)\| < 1$ and hence, we have $\|Tx\| < 2/\delta$. Part (iii) follows.
Finally, we need to show $(iii) \Rightarrow (ii)$. Note that by the assumption of (iii), there is $M > 0$ such that $\|Tx\| \leq M$ for all $x \in B_X$. Thus, for each $x \in X$, we have $\|Tx\| \leq M\|x\|$. This implies that T is continuous at 0. The proof is complete. □

Corollary 3.2. *Let $T : X \to Y$ be a bounded linear map. Then we have*

$$\sup\{\|Tx\| : x \in B_X\} = \sup\{\|Tx\| : x \in S_X\} = \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}.$$

Proof. Let $a = \sup\{\|Tx\| : x \in B_X\}$, $b = \sup\{\|Tx\| : x \in S_X\}$ and $c = \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}$.
Clearly, we have $b \leq a$. Now for each $x \in B_X$ with $x \neq 0$, then we have $b \geq \|T(x/\|x\|)\| = (1/\|x\|)\|Tx\| \geq \|Tx\|$. Thus, we have $b \geq a$ and thus, $a = b$.
Now if $M > 0$ satisfies $\|Tx\| \leq M\|x\|$, $\forall x \in X$, then we have $\|Tw\| \leq M$ for all $w \in S_X$. Hence, we have $b \leq M$ for all such M, and so we have $b \leq c$. Finally, it remains to show $c \leq b$. Note that by the definition of b, we have $\|Tx\| \leq b\|x\|$ for all $x \in X$. Thus, $c \leq b$. □

Proposition 3.3. *Let X and Y be normed spaces. Let $B(X, Y)$ be the set of all bounded linear maps from X into Y. For each element $T \in B(X, Y)$, let*

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

be defined as in Proposition 3.1.
Then $(B(X, Y), \|\cdot\|)$ becomes a normed space.
Furthermore, if Y is a Banach space, then so is $B(X, Y)$.
In particular, if $Y = \mathbb{K}$, then $B(X, \mathbb{K})$ is a Banach space. In this case, put $X^* := B(X, \mathbb{K})$ and call it the **dual space** *of X.*

Proof. We can directly check that $B(X, Y)$ is a normed space (**Do It By Yourself!**).
We want to show that $B(X, Y)$ is complete if Y is a Banach space. Let (T_n) be a Cauchy sequence in $B(X, Y)$. Then for each $x \in X$, it is easy to see that $(T_n x)$ is a Cauchy sequence in Y. Thus, $\lim T_n x$ exists in Y for each $x \in X$ because Y is complete. Hence, we can define a map $Tx := \lim T_n x \in Y$ for each $x \in X$. Clearly, T is a linear map from X into Y.
We need show that $T \in B(X, Y)$ and $\|T - T_n\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since (T_n) is a Cauchy sequence in $B(X, Y)$, there is a positive integer N such that $\|T_m - T_n\| < \varepsilon$ for all $m, n \geq N$. Hence, we have $\|(T_m - T_n)(x)\| < \varepsilon$ for all $x \in B_X$ and $m, n \geq N$. Taking $m \to \infty$, we have $\|Tx - T_n x\| \leq \varepsilon$ for all $n \geq N$ and $x \in B_X$. Therefore, we have $\|T - T_n\| \leq \varepsilon$ for all $n \geq N$. From this, we see that $T - T_N \in B(X, Y)$ and thus, $T = T_N + (T - T_N) \in B(X, Y)$ and $\|T - T_n\| \to 0$ as $n \to \infty$. Therefore, $\lim_n T_n = T$ exists in $B(X, Y)$. □

Remark 3.4. *By using Proposition 3.1, we can show that if $f : X \to \mathbb{K}$ is any linear functional defined on a vector space X, then X can be endowed with a norm so that f is bounded.*
In fact, if we fix a vector basis $(e_i)_{i\in I}$ for X and put $\|x\|_\infty := \max_{i\in I} |a_i|$ as $x = \sum_{i\in I} a_i e_i \in X$, (note that it is a finite sum), where $a_i \in \mathbb{K}$, then the function $\|\cdot\|_\infty$ is a norm on X. Now for each $x \in X$, set

$$\|x\|_1 := |f(x)| + \|x\|_\infty.$$

Clearly, the function $\|\cdot\|_1$ is a norm on X. In addition, we have $|f(x)| \le \|x\|_1$ for all $x \in X$. Hence, f is bounded on X with respect to the norm $\|\cdot\|_1$ as required.

Proposition 3.5. *Let X and Y be normed spaces. Suppose that X is of finite dimension n. Then we have the following assertions.*

(i) Any linear operator from X into Y must be bounded.
(ii) If $T_k : X \to Y$ is a sequence of linear operators such that $T_k x \to 0$ for all $x \in X$, then $\|T_k\| \to 0$.

Proof. Using Proposition 2.3 and the notations as in the proof, then there is $c > 0$ such that

$$\sum_{i=1}^{n} |\alpha_i| \le c\|\sum_{i=1}^{n} \alpha_i e_i\|$$

for all scalars $\alpha_1, ..., \alpha_n$. Therefore, for any linear map T from X to Y, we have

$$\|Tx\| \le \big(\max_{1\le i\le n} \|Te_i\|\big)c\|x\|$$

for all $x \in X$. This gives the assertions (i) and (ii) immediately. □

Remark 3.6. *The assumption of X of finite dimension in Proposition 3.5 cannot be removed. For example, if for each positive integer k, we define $f_k : c_0 \to \mathbb{R}$ by $f_k(x) := x(k)$, then f_k is bounded for each k and*

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} x(k) = 0$$

for all $x \in c_0$. However $f_k \nrightarrow 0$ because $\|f_k\| \equiv 1$ for every k.

Proposition 3.7. *Let Y be a closed subspace of X and X/Y be the quotient space. For each element $x \in X$, put $\bar{x} := x + Y \in X/Y$ the corresponding element in X/Y. Define*

$$\|\bar{x}\| = \inf\{\|x + y\| : y \in Y\}. \tag{3.1}$$

If we let $\pi : X \to X/Y$ be the natural projection, i.e., $\pi(x) = \bar{x}$ for all $x \in X$, then $(X/Y, \|\cdot\|)$ is a normed space and π is bounded with $\|\pi\| \le 1$. In particular, $\|\pi\| = 1$ as Y is a proper closed subspace.
Furthermore, if X is a Banach space, then so is X/Y.
In this case, we call $\|\cdot\|$ in (3.1) the quotient norm on X/Y.

Proof. Note that since Y is closed, we can directly check that $\|\bar{x}\| = 0$ if and only is $x \in Y$, i.e., $\bar{x} = \bar{0} \in X/Y$. It is easy to check the other conditions of the definition of a norm. Thus, X/Y is a normed space. Moreover, π is clearly bounded with $\|\pi\| \le 1$ by the definition of the quotient norm on X/Y.
Furthermore, if $Y \subsetneq X$, then by using the Riesz's Lemma 2.5, we see that $\|\pi\| = 1$.
We show the last assertion. Suppose that X is a Banach space. Let $(\bar{x}_n)$ be a Cauchy sequence in X/Y. It suffices to show that $(\bar{x}_n)$ has a convergent subsequence in X/Y.
Indeed, since $(\bar{x}_n)$ is a Cauchy sequence, we can find a subsequence $(\bar{x}_{n_k})$ of $(\bar{x}_n)$ such that

$$\|\bar{x}_{n_{k+1}} - \bar{x}_{n_k}\| < 1/2^k$$

for all $k = 1, 2....$ Then by the definition of quotient norm, there is an element $y_1 \in Y$ such that $\|x_{n_2} - x_{n_1} + y_1\| < 1/2$. Note that we have, $\overline{x_{n_1} - y_1} = \bar{x}_{n_1}$ in X/Y. Thus, there is $y_2 \in Y$ such that $\|x_{n_2} - y_2 - (x_{n_1} - y_1)\| < 1/2$ by the definition of quotient norm again. In addition, we have $\overline{x_{n_2} - y_2} = \bar{x}_{n_2}$. Then we also have an element $y_3 \in Y$ such that $\|x_{n_3} - y_3 - (x_{n_2} - y_2)\| < 1/2^2$. To repeat the same step, we can obtain a sequence (y_k) in Y such that

$$\|x_{n_{k+1}} - y_{k+1} - (x_{n_k} - y_k)\| < 1/2^k$$

for all $k = 1, 2....$ Therefore, $(x_{n_k} - y_k)$ is a Cauchy sequence in X and thus, $\lim_k (x_{n_k} - y_k)$ exists in X while X is a Banach space. Set $x = \lim_k (x_{n_k} - y_k)$. On the other hand, note that we have $\pi(x_{n_k} - y_k) = \pi(x_{n_k})$ for all $k = 1, 2, , ,$. This tells us that $\lim_k \pi(x_{n_k}) = \lim_k \pi(x_{n_k} - y_k) = \pi(x) \in X/Y$ since π is bounded. Therefore, $(\bar{x}_{n_k})$ is a convergent subsequence of $(\bar{x}_n)$ in X/Y. The proof is complete. □

Corollary 3.8. *Let $T : X \to Y$ be a linear map. Suppose that Y is of finite dimension. Then T is bounded if and only if* $\ker T := \{x \in X : Tx = 0\}$ *is closed.*

Proof. The necessary part is clear.
Now assume that $\ker T$ is closed. Then by Proposition 3.7, $X/\ker T$ becomes a normed space. Morover, it is known that there is a linear injection $\widetilde{T} : X/\ker T \to Y$ such that $T = \widetilde{T} \circ \pi$, where $\pi : X \to X/\ker T$ is the natural projection. Since $\dim Y < \infty$ and $\widetilde{T}$ is injective, $\dim X/\ker T < \infty$. This implies that $\widetilde{T}$ is bounded by Proposition 3.5. Hence T is bounded because $T = \widetilde{T} \circ \pi$ and π is bounded. □

Remark 3.9. The converse of Corollary 3.8 does not hold when Y is of infinite dimension. For example, let $X := \{x \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x(n)|^2 < \infty\}$ (note that X is a vector space **Why?**) and $Y = \ell^2$. Both X and Y are endowed with $\|\cdot\|_2$-norm.
Define $T : X \to Y$ by $Tx(n) = nx(n)$ for $x \in X$ and $n = 1, 2....$ Then T is an unbounded operator(**Check !!**). Note that $\ker T = \{0\}$ and hence, $\ker T$ is closed. Hence, the closeness of $\ker T$ does not imply the boundedness of T in general.

Two normed spaces X and Y are said to be *isomorphic (resp. isometric isomorphic)* if there is a bi-continuous linear isomorphism (resp. isometric) between X and Y. We write $X = Y$ if X and Y are isometric isomorphic.

Remark 3.10. *Note that the inverse of a bounded linear isomorphism need not be bounded.*

Example 3.11. *Let $X : \{f \in C^\infty(-1, 1) : f^{(n)} \in C^b(-1, 1)$ for all $n = 0, 1, 2...\}$ and $Y := \{f \in X : f(0) = 0\}$. In addition, X and Y both are equipped with the* sup-*norm $\|\cdot\|_\infty$. Define an operator $S : X \to Y$ by*

$$Sf(x) := \int_0^x f(t)dt$$

for $f \in X$ and $x \in (-1, 1)$. Then S is a bounded linear isomorphism but its inverse S^{-1} is unbounded. In fact, the inverse $S^{-1} : Y \to X$ is given by

$$S^{-1}g := g'$$

for $g \in Y$.

A metric space is said to be *separable* if there is a countable dense subset, for example, the base field $\mathbb{K}$ is separable. Moreover, it is easy to see that a normed space is separable if and only if it is the closed linear span of a countable dense subset.

Definition 3.12. *A sequence of element* $(e_n)_{n=1}^{\infty}$ *in a normed space* X *is called a Schauder basis for* X *if for each element* $x \in X$*, there is a unique sequence of scalars* (α_n) *such that*

$$x = \sum_{n=1}^{\infty} \alpha_n e_n. \tag{3.2}$$

Note: The expression in Eq. 3.2 depends on the order of e_n**'s.**

Remark 3.13. Note that if X has a Scahuder basis, then X must be separable. The following natural question was first raised by Banach (1932).
The basis problem: Does every separable Banach space have a Schauder basis?
The answer is "**No**$''$!
This problem was completely solved by P. Enflo in 1973.

Example 3.14. *We have the following assertions.*

(i) *The space* ℓ^∞ *is non-separable under the* sup*-norm* $\|\cdot\|_\infty$*. Consequently,* ℓ^∞ *has no Schauder basis.*
(ii) *The spaces* c_0 *and* ℓ^p *for* $1 \leq p < \infty$ *have Schauder bases.*

Proof. For Part (i) let $D = \{x \in \ell^\infty : x(i) = 0 \text{ or } 1\}$. Then D is an uncountable set and $\|x - y\|_\infty = 1$ for $x \neq y$. Therefore $\{B(x, 1/4) : x \in D\}$ is an uncountable family of disjoint open balls. Therefore, ℓ^∞ has no countable dense subset.
For each $n = 1, 2...$, let $e_n(i) = 1$ if $n = i$, otherwise, is equal to 0.
In addition, (e_n) is a Schauder basis for the space c_0 and ℓ^p for $1 \leq p < \infty$. □

In the rest of this section, we are going to investigate some concrete examples of dual spaces.

Example 3.15. *Let* $X = \mathbb{K}^N$*. Consider the usual Euclidean norm on* X*, i.e.,* $\|(x_1, ..., x_N)\| := \sqrt{|x_1|^2 + \cdots |x_N|^2}$*. Define* $\theta : \mathbb{K}^N \to (\mathbb{K}^N)^*$ *by* $\theta x(y) = x_1 y_1 + \cdots + x_N y_N$ *for* $x = (x_1, ..., x_N)$ *and* $y = (y_1, ..., y_N) \in \mathbb{K}^N$*. Note that* $\theta x(y) = \langle x, y \rangle$*, the usual inner product on* $\mathbb{K}^N$*. Then by the Cauchy-Schwarz inequality, it is easy to see that* θ *is an isometric isomorphism. Therefore, we have* $\mathbb{K}^N = (\mathbb{K}^N)^*$*.*

Example 3.16. *Define a map* $T : \ell^1 \to c_0^*$ *by*

$$(Tx)(\eta) = \sum_{i=1}^{\infty} x(i)\eta(i)$$

for $x \in \ell^1$ *and* $\eta \in c_0$*.*
Then T *is isometric isomorphism and hence,* $c_0^* = \ell^1$*.*

Proof. The proof is divided into the following steps.
Step 1. $Tx \in c_0^*$ for all $x \in \ell^1$.
In fact, let $\eta \in c_0$. Then

$$|Tx(\eta)| \leq |\sum_{i=1}^{\infty} x(i)\eta(i)| \leq \sum_{i=1}^{\infty} |x(i)||\eta(i)| \leq \|x\|_1 \|\eta\|_\infty.$$

Step 1 follows.
Step 2. T is an isometry.
Note that by *Step* 1, we have $\|Tx\| \leq \|x\|_1$ for all $x \in \ell^1$. We need to show that $\|Tx\| \geq \|x\|_1$ for

all $x \in \ell^1$. Fix $x \in \ell^1$. Now for each $k = 1, 2..$, consider the polar form $x(k) = |x(k)|e^{i\theta_k}$. Note that $\eta_n := (e^{-i\theta_1}, ..., e^{-i\theta_n}, 0, 0, ...) \in c_0$ for all $n = 1, 2....$ Then we have

$$\sum_{k=1}^{n} |x(k)| = \sum_{k=1}^{n} x(k)\eta_n(k) = Tx(\eta_n) = |Tx(\eta_n)| \leq \|Tx\|$$

for all $n = 1, 2....$ Hence, we have $\|x\|_1 \leq \|Tx\|$.

Step 3. T is a surjection.

Let $\phi \in c_0^*$ and let $e_k \in c_0$ be given by $e_k(j) = 1$ if $j = k$, otherwise, is equal to 0. Put $x(k) := \phi(e_k)$ for $k = 1, 2...$ and consider the polar form $x(k) = |x(k)|e^{i\theta_k}$ as above. Then we have

$$\sum_{k=1}^{n} |x(k)| = \phi(\sum_{k=1}^{n} e^{-i\theta_k} e_k) \leq \|\phi\| \|\sum_{k=1}^{n} e^{-i\theta_k} e_k\|_\infty = \|\phi\|$$

for all $n = 1, 2....$ Therefore, $x \in \ell^1$.

Finally, we need to show that $Tx = \phi$ and thus, T is surjective. In fact, if $\eta = \sum_{k=1}^{\infty} \eta(k)e_k \in c_0$, then we have

$$\phi(\eta) = \sum_{k=1}^{\infty} \eta(k)\phi(e_k) = \sum_{k=1}^{\infty} \eta(k)x(k) = Tx(\eta).$$

The proof is complete by the *Steps* 1-3 above. □

Example 3.17. We have the other important examples of the dual spaces.

(i) $(\ell^1)^* = \ell^\infty$.
(ii) For $1 < p < \infty$, $(\ell^p)^* = \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.
(iii) For a locally compact Hausdorff space X, $C_0(X)^* = M(X)$, where $M(X)$ denotes the space of all regular Borel measures on X.

Parts (i) and (ii) can be obtained by the similar argument as in Example 3.16 (see also in [**?**, Chapter 8]). Part (iii) is known as the *Riesz representation Theorem* which is referred to [**?**, Section 21.5] for the details.

Example 3.18. Let $C[a, b]$ be the space of all continuous $\mathbb{R}$-valued functions defined on a closed and bounded interval $[a, b]$. Moreover, the space $C[a, b]$ is endowed with the sup-norm, i.e., $\|f\|_\infty := \sup\{|f(x)| : x \in [a, b]$ for $f \in C[a, b]$.

A function $\rho : [a, b] \to \mathbb{R}$ is said to be a bounded variation if it satisfies the condition:

$$V(\rho) := \sup\{\sum_{k=1}^{n} |\rho(x_k) - \rho(x_{k-1})| : a = x_0 < x_1 < \cdots < x_n = b\} < \infty.$$

Let $BV([a, b])$ denote the space of all bounded variations on $[a, b]$ and let $\|\rho\| := V(\rho)$ for $\rho \in BV([a, b])$. Then $BV([a, b])$ becomes a Banach space.

Besides, for $f \in C[a, b]$, the Riemann-Stieltjes integral of f with respect to a bounded variation ρ on $[a, b]$ is defined by

$$\int_a^b f(x)d\rho(x) := \lim_P \sum_{k=1}^{n} f(\xi_k)(\rho(x_k) - \rho(x_{k-1})),$$

where $P : a = x_0 < x_1 < \cdots < x_n = b$ and $\xi_k \in [x_{k-1}, x_k]$ (**Fact: the Riemann-Stieltjes integral of a continuous function always exists**).

Define a mapping $T : BV([a, b]) \to C[a, b]^*$ by

$$T(\rho)(f) := \int_a^b f(x)d\rho(x)$$

for $\rho \in BV([a,b])$ and $f \in C[a,b]$. Then T is an isometric isomorphism, and hence, we have

$$C[a,b]^* = BV([a,b]).$$

4. Hahn-Banach Theorem

A real valued function $p : X \to \mathbb{R}$ defined on a vector space X is called a *positively homogeneous sub-additive* if the following conditions hold:

(i) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \geq 0$.
(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Lemma 4.1. *Let X be a real vector space and Y be a subspace of X. Assume that there is an element $v \in X \setminus Y$ such that $X = Y \oplus \mathbb{R}v$, i.e., the space X is the linear span of Y and v. Let p be a positive homogeneous sub-additive function defined on X. Suppose that f is real linear functional defined on Y satisfying $f(y) \leq p(y)$ for all $y \in Y$. Then there is a real linear extension F of f defined on X so that*

$$F(x) \leq p(x) \qquad \textit{for all } x \in X.$$

Proof. It is noted that if F is a linear extension of f on X and $\gamma := F(v)$ which satisfies

$$F(y+tv) = f(y) + t\gamma \leq p(y+tv) \quad \text{for all } y \in Y \text{ and for all } t \in \mathbb{R},$$

then it suffices to saying that the following inequalities hold:

$$f(y_1) + \gamma \leq p(y_1 + v) \quad \text{and} \quad f(y_2) - \gamma \leq p(y_2 - v) \tag{4.1}$$

for all $y_1, y_2 \in Y$. Thus, we need to determine $\gamma := F(v)$ so that the following holds:

$$f(y_1) - p(y_1 - v) \leq \gamma \leq -f(y_2) + p(y_2 + v) \quad \text{for all } y_1, y_2 \in Y. \tag{4.2}$$

Note that if we fix $y_1, y_2 \in Y$, we see that

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - v) + p(y_2 + v).$$

This implies that we have

$$f(y_1) - p(y_1 - v) \leq -f(y_2) + p(y_2 + v)$$

for all $y_1, y_2 \in Y$. Therefore, it gives

$$a := \sup\{f(y_1) - p(y_1 - v) : y_1 \in Y\} \quad \leq \quad b := \inf\{-f(y_2) + p(y_2 + v) : y_2 \in Y\}.$$

Now if we choose a real number γ so that $a \leq \gamma \leq b$, then the Inequality 4.2 holds. The proof is complete. □

Remark 4.2. Before completing the proof of the Hahn-Banach Theorem, Let us first recall one of super important results in mathematics, called *Zorn's Lemma*, a very humble name. Every mathematics student should know it.

Zorn's Lemma: Let $\mathcal{X}$ be a non-empty set with a partially order " $\leq$ ". Assume that every totally order subset $\mathcal{C}$ of $\mathcal{X}$ has an upper bound, i.e. there is an element $\mathfrak{z} \in \mathcal{X}$ such that $c \leq \mathfrak{z}$ for all $c \in \mathcal{C}$. Then $\mathcal{X}$ must contain a maximal element $\mathfrak{m}$, that is, if $\mathfrak{m} \leq x$ for some $x \in \mathcal{X}$, then $\mathfrak{m} = x$.

The following is the typical argument of applying the Zorn's Lemma.

Theorem 4.3. Hahn-Banach Theorem : *Let X be a vector space (not necessary to be a normed space) over $\mathbb{R}$ and let Y be a subspace of X. Let p be a positive homogeneous sub-additive function* *defined on X. Suppose that f is a real linear functional defined on Y satisfying $f(y) \leq p(y)$ for all $y \in Y$. Then there is a real linear extension F of f defined on X so that*

$$F(x) \leq p(x) \qquad \textit{for all } x \in X.$$

Proof. Let $\mathfrak{X}$ be the collection of the pairs (Y_1, f_1), where $Y \subseteq Y_1$ is a subspace of X and f_1 is a linear extension of f defined on Y_1 such that and $f_1 \leq p$ on Y_1. Define a partial order $\leq$ on $\mathfrak{X}$ by $(Y_1, f_1) \leq (Y_2, f_2)$ if $Y_1 \subseteq Y_2$ and $f_2|_{Y_1} = f_1$. Then by the Zorn's lemma, there is a maximal element $(\widetilde{Y}, F)$ in $\mathfrak{X}$. The maximality of $(\widetilde{Y}, F)$ and Lemma 4.1 give $\widetilde{Y} = X$. The proof is complete. □

Definition 4.4. *Let D be a convex subset of a normed space X, i.e., $tx + (1-t)y \in D$ for all $x, y \in D$ and $t \in (0,1)$. Suppose that 0 is an interior point of D. Define*

$$\mu_D(x) := \inf\{t > 0 : x \in tD\}$$

for $x \in X$. In addition, set $\mu_D(x) = \infty$ if $\{t > 0 : x \in tD\} = \emptyset$.
The function μ_D is called the Minkowski functional with respect to D.

Lemma 4.5. *Let D be a convex subset of a normed space X. Suppose that 0 is an interior point of D. Then the Minkowski functional $\mu := \mu_D : X \to [0, \infty)$ is positively homogeneous and sub-additive on D.*
In addition, we have $\{x \in X : \mu(x) < 1\} \subseteq D \subseteq \{x \in X : \mu(x) \leq 1\}$.

Proof. It is noted that since $0 \in int(D)$, the set $\{t > 0 : x \in tD\} \neq \emptyset$ for all $x \in X$. Thus, the function $\mu : X \to [0, \infty)$ is defined.
Clearly, if we fix $t > 0$ and $x \in X$, then we have $\mu(tx) \leq s$ if and only if $t\mu(x) \leq s$. Hence, the function μ is positively homogeneous.
Next, we show the subadditivity of μ. Let $\varepsilon > 0$. For $x, y \in X$, we choose $s, t > 0$ such that $x \in sD$ and $y \in tD$ satisfying $s < \mu(x) + \varepsilon$ and $t < \mu(y) + \varepsilon$. Then $x = sd_1$ and $y = td_2$ for some $d_1, d_2 \in D$. Since D is convex, we have

$$x + y = sd_1 + td_2 = (s+t)(\frac{s}{s+t}d_1 + \frac{t}{s+t}d_2) \in (s+t)D.$$

Thus, $\mu(x+y) \leq s+t$ and so, $\mu(x+y) < \mu(x) + \mu(y) + 2\varepsilon$. Therefore, μ is sub-additive. The last assertion is clear by the definition of μ. □

Proposition 4.6. *Let C be a closed convex subset of a real vector space X. Let d and A be the positive constants. If x_0 is an element in X with $\|x_0\| \leq A$ such that $0 < d < dist(x_0, C)$, then there is an element $F_1 \in B_{X^*}$ such that*

$$F_1(y) + \alpha < F_1(x_0). \tag{4.3}$$

for all $y \in C$, whenever $0 < \alpha < \frac{d}{2}(1 - \frac{d}{2A})^{-1}$.
Consequently, for any closed convex subset C_1 of X and $x_0' \notin C_1$, then there is an element $g \in X^*$ with $\|g\| \leq 1$ such that

$$g(z) + \beta < g(x_0'). \tag{4.4}$$

for all $z \in C_1$, whenever $0 < \beta < dist(x_0', C_1)$.

Proof. For showing the first assertion, we first note that if there is $x_0 \in X$ such that $\|x_0\| \leq A$ and $d(x_0, C) > d$. then $\frac{d}{2A} < 1$. To see this, we have $A \geq \|x_0\| \geq d(x_0, C) > \frac{d}{2}$ because $0 \in C$.
Now since $0 < d < dist(x_0, C)$, we have $(x_0 + B(0, d)) \cap C = \emptyset$. Thus, we have $(x_0 + B(0, \frac{1}{2}d)) \cap (C + B(0, \frac{1}{2}d)) = \emptyset$. Put $D := C + B(0, \frac{1}{2}d)$. Notice that D is a convex subset of X and $x_0 \notin D$. Moreover, we have $0 \in int(D)$. Let $\mu := \mu_D$ be the Minkowski functional corresponding to D. Then μ is positive homogeneous and sub-additive on X by Lemma 4.5.

Put $Y := \mathbb{R}x_0$ and define $f : Y \to \mathbb{R}$ by $f(\alpha x_0) := \alpha\mu(x_0)$ for $\alpha \in \mathbb{R}$. Then $f(y) \leq \mu(y)$ for all $y \in Y$ since $\mu \geq 0$ and positive homogenous. The Hahn-Banch Theorem 4.3 implies that there is a linear extension F defined on X satisfying $F(x) \leq \mu(x)$ for all $x \in X$. We want to show that the linear functional $F_1 := \frac{d}{2}F \in B_{X^*}$ is as required.
We first notice that F is bounded because we have $|F(y)| \leq \mu(y) \leq 1$ for all $y \in B(0, \frac{1}{2}d) \subseteq D$ and so, $\|F_1\| = \|\frac{d}{2}F\| \leq 1$. Note that $\mu(x) \leq 1$ for all $x \in C$ because $C \subseteq D$. Thus, $\sup F(C) \leq 1$. On the other hand, since $x_0 \notin D$, we have $F(x_0) = \mu(x_0) \geq 1$. Now if $\mu(x_0) = 1$, then there is a decreasing sequence of positive numbers (λ_n) with $\lambda_n \downarrow 1$ and $\frac{1}{\lambda_n}x_0 \in D$. This implies that $x_0 \in \overline{D}$. It contradicts to the fact that $(x_0 + B(0, \frac{1}{2}d)) \cap D$ is empty. Hence, we have $F(y) \leq 1 < F(x_0) = \mu(x_0)$ for all $y \in D$.
Next, we are going to show that the Inequality 4.3 holds. In fact, for $\lambda > 0$, we see that $x_0 \in \lambda D$ if and only if $\frac{1}{\lambda}x_0 \in D$. Hence, In this case, we have $1 < \mu(x_0) \leq \lambda$. Also, we have $\frac{1}{\lambda}x_0 = y + z$ for some $y \in C$ and for some $z \in B(0, \frac{1}{2}d)$. Then we have

$$d \leq \|x_0 - y\| = \|x_0 - \frac{1}{\lambda}x_0 - z\| = |1 - \frac{1}{\lambda}|\|x_0\| + \|z\| < |1 - \frac{1}{\lambda}|A + \frac{1}{2}d.$$

This implies that $1 - \frac{1}{\lambda} > \frac{d}{2A}$ because $\lambda \geq \mu(x_0) > 1$. This gives $1 < (1 - \frac{d}{2A})^{-1} < \lambda$ whenever $\lambda > 0$ with $x_0 \in \lambda D$ and hence, $\mu(x_0) \geq (1 - \frac{d}{2A})^{-1}$. Now if we put $0 < \alpha_1 := (1 - \frac{d}{2A})^{-1} - 1$, then we have

$$F(y) + \alpha_1 \leq 1 + \alpha_1 < (1 - \frac{d}{2A})^{-1} \leq \mu(x_0) = F(x_0)$$

for all $y \in C$. Therefore, if $0 < \alpha < \frac{d}{2}\alpha_1$, then the element $F_1 := \frac{d}{2}F \in B_{X^*}$ satisfies the inequality 4.3 as desired.
For showing the last assertion, let $d_0 := dist(x_0', C_1) > 0$. Clearly, we have $\lim\limits_{t \to 1-} \dfrac{td_0}{2}(1 - \dfrac{td_0}{2d_0})^{-1} = d_0$. Thus if $0 < \beta < d_0$, there is $0 < t_1 < 1$ such that $\beta < \frac{t_1 d_0}{2}(1 - \frac{t_1 d_0}{2d_0})^{-1}$. Now if we put $d := t_1 d_0$, then we have $0 < \beta < \frac{d}{2}(1 - \frac{d}{2d_0})^{-1}$. From this, we choose $\varepsilon > 0$ such that $\beta < \frac{d}{2}(1 - \frac{d}{2(d_0+\varepsilon)})^{-1}$. Now we fix a point $x_1 \in C_1$ so that $\|x_0' - x_1\| < d_0 + \varepsilon$. Put $x_0 := x_0' - x_1$; $C := C_1 - x_1$ and $A := d_0 + \varepsilon$ into the first assertion. Then there is an element $g \in B_{X^*}$ such that the inequality 4.4 holds immediately.
We finish the proof. □

The following result is also referred to the **Hahn-Banach Theorem.**

Theorem 4.7. *Let X be a normed space and let Y be a subspace of X. If $f \in Y^*$, then there exists a linear extension $F \in X^*$ of f such that $\|F\| = \|f\|$.*

Proof. W.L.O.G, we may assume that $\|f\| = 1$. We first show the case when X is normed space over $\mathbb{R}$. It is noted that the norm function $p(\cdot) := \|\cdot\|$ is positively homogeneous and sub-additive on X. Since $\|f\| = 1$, we have $f(y) \leq p(y)$ for all $y \in Y$. Then by the Hahn-Banach Theorem 4.3, there is a linear extension F of f on X such that $F(x) \leq p(x)$ for all $x \in X$. This implies that $\|F\| = 1$ as required.
Now for the complex case, let $h = \Re ef$ and $g = \Im mf$. Then $f = h + ig$ and f, g both are real linear on Y with $\|h\| \leq 1$. Note that since $f(iy) = if(y)$ for all $y \in Y$, we have $g(y) = -h(iy)$ for all $y \in Y$. This gives $f(\cdot) = h(\cdot) - ih(i\cdot)$ on Y. Then by the real case above, there is a real linear extension H on X such that $\|H\| = \|h\|$. Now define $F : X \longrightarrow \mathbb{C}$ by $F(\cdot) := H(\cdot) - iH(i\cdot)$. Then $F \in X^*$ and $F|_Y = f$. Thus it remains to show that $\|F\| = \|f\| = 1$. We need to show that $|F(z)| \leq \|z\|$ for all $z \in X$. For $z \in X$, consider the polar form $F(z) = re^{i\theta}$. Then $F(e^{-i\theta}z) = r \in \mathbb{R}$ and thus $F(e^{-i\theta}z) = H(e^{-i\theta}z)$. This yields that

$$|F(z)| = r = |F(e^{-i\theta}z)| = |H(e^{-i\theta}z)| \leq \|H\|\|e^{-i\theta}z\| \leq \|z\|.$$

The proof is complete. □

Proposition 4.8. *Let X be a normed space and $x_0 \in X$. Then there is $f \in X^*$ with $\|f\| = 1$ such that $f(x_0) = \|x_0\|$. Consequently, we have*

$$\|x_0\| = \sup\{|g(x)| : g \in B_{X^*}\}.$$

In addition, if $x, y \in X$ with $x \neq y$, then there exists $f \in X^*$ such that $f(x) \neq f(y)$.

Proof. Let $Y = \mathbb{K}x_0$. Define $f_0 : Y \to \mathbb{K}$ by $f_0(\alpha x_0) := \alpha\|x_0\|$ for $\alpha \in \mathbb{K}$. Then $f_0 \in Y^*$ with $\|f_0\| = \|x_0\|$. The result follows immediately from the Hahn-Banach Theorem. □

Remark 4.9. Proposition 4.8 tells us that the dual space X^* of X must be non-zero. Indeed, the dual space X^* is very "*Large*" so that it can separate any pair of distinct points in X.
Furthermore, for any normed space Y and any pair of points $x_1, x_2 \in X$ with $x_1 \neq x_2$, we can find an element $T \in B(X, Y)$ such that $Tx_1 \neq Tx_2$. In fact, fix a non-zero element $y \in Y$. Then by Proposition 4.8, there is $f \in X^*$ such that $f(x_1) \neq f(x_2)$. Thus, if we define $Tx = f(x)y$, then $T \in B(X, Y)$.

Proposition 4.10. *Using the notations as above, if M is closed subspace and $v \in X \setminus M$, then there is $f \in X^*$ such that $f(M) \equiv 0$ and $f(v) \neq 0$.*

Proof. Since M is a closed subspace of X, we can consider the quotient space X/M. Let $\pi : X \to X/M$ be the natural projection. Note that $\bar{v} := \pi(v) \neq 0 \in X/M$ because $\bar{v} \in X \setminus M$. Then by Corollary 4.8, there is a non-zero element $\bar{f} \in (X/M)^*$ such that $\bar{f}(\bar{v}) \neq 0$. Therefore, the linear functional $f := \bar{f} \circ \pi \in X^*$ is as desired. □

Proposition 4.11. *Using the notations as above, if X^* is separable, then X is separable.*

Proof. Let $F := \{f_1, f_2....\}$ be a dense subset of X^*. Then there is a sequence (x_n) in X with $\|x_n\| = 1$ and $|f_n(x_n)| \geq 1/2\|f_n\|$ for all n. Now let M be the closed linear span of x_n's. Then M is a separable closed subspace of X. We are going to show that $M = X$. Suppose that $M \neq X$ and hence Proposition 4.10 gives us a non-zero element $f \in X^*$ such that $f(M) \equiv 0$. Since $\{f_1, f_2....\}$ is dense in X^*, we have $B(f, r) \cap F \neq \emptyset$ for all $r > 0$. Therefore, if $B(f, r) \cap F \neq \emptyset$ is finite for some $r > 0$, then $f = f_m$ for some $f_m \in F$. This implies that $\|f\| = \|f_m\| \leq 2|f_m(x_m)| = 2|f(x_m)| = 0$ and thus, $f = 0$ which contradicts to $f \neq 0$.
Therefore, $B(f, r) \cap F$ is infinite for all $r > 0$. In this case, there is a subsequence (f_{n_k}) such that $\|f_{n_k} - f\| \to 0$. This gives

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \to 0$$

because $f(M) \equiv 0$. Thus$\|f_{n_k}\| \to 0$ and hence $f = 0$. It leads to a contradiction again. Thus, we can conclude that $M = X$ as desired. □

Remark 4.12. The converse of Proposition 4.11 does not hold. For example, consider $X = \ell^1$. Then ℓ^1 is separable but the dual space $(\ell^1)^* = \ell^\infty$ is not.

Proposition 4.13. *Let X and Y be normed spaces. For each element $T \in B(X, Y)$, define a linear operator $T^* : Y^* \to X^*$ by*

$$T^*y^*(x) := y^*(Tx)$$

for $y^* \in Y^*$ and $x \in X$. Then $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$. In this case, T^* is called the adjoint operator of T.

Proof. We first claim that $\|T^*\| \le \|T\|$ and hence, $\|T^*\|$ is bounded.
In fact, for any $y^* \in Y^*$ and $x \in X$, we have $|T^*y^*(x)| = |y^*(Tx)| \le \|y^*\|\|T\|\|x\|$. Hence, $\|T^*y^*\| \le \|T\|\|y^*\|$ for all $y^* \in Y^*$. Thus, $\|T^*\| \le \|T\|$.
We need to show $\|T\| \le \|T^*\|$. Let $x \in B_X$. Then by Proposition 4.8, there is $y^* \in S_{X^*}$ such that $\|Tx\| = |y^*(Tx)| = |T^*y^*(x)| \le \|T^*y^*\| \le \|T^*\|$. This implies that $\|T\| \le \|T^*\|$. □

Example 4.14. Let X and Y be the finite dimensional normed spaces. Let $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^m$ be the bases for X and Y respectively. Let $\theta_X : X \to X^*$ and $\theta_Y : X \to Y^*$ be the identifications as in Example 3.15. Let $e_i^* := \theta_X e_i \in X^*$ and $f_j^* := \theta_Y f_j \in Y^*$. Then $e_i^*(e_l) = \delta_{il}$ and $f_j^*(f_l) = \delta_{jl}$, where, $\delta_{il} = 1$ if $i = l$; otherwise is 0.
Now if $T \in B(X,Y)$ and $(a_{ij})_{m\times n}$ is the representative matrix of T corresponding to the bases $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^m$ respectively, then $a_{kl} = f_k^*(Te_l) = T^*f_k^*(e_l)$. Therefore, if $(a'_{lk})_{n\times m}$ is the representative matrix of T^* corresponding to the bases (f_j^*) and (e_i^*), then $a_{kl} = a'_{lk}$. Hence the transpose $(a_{kl})^t$ is the the representative matrix of T^*.

Proposition 4.15. *Let Y be a closed subspace of a normed space X. Let $i : Y \to X$ be the natural inclusion and $\pi : X \to X/Y$ the natural projection. Then*

(i) the adjoint operator $i^{**} : Y^{**} \to X^{**}$ is an isometry.
(ii) the adjoint operator $\pi^* : (X/Y)^* \to X^*$ is an isometry.

Consequently, Y^{**} and $(X/Y)^*$ can be viewed as the closed subspaces of X^{**} and X^* respectively.

Proof. For Part (i), we first note that for any $x^* \in X^*$, the image i^*x^* in Y^* is just the restriction of x^* on Y, denoted by $x^*|_Y$. Now let $\phi \in Y^{**}$. Then for any $x^* \in X^*$, we have

$$|i^{**}\phi(x^*)| = |\phi(i^*x^*)| = |\phi(x^*|_Y)| \le \|\phi\|\|x^*|_Y\|_{Y^*} \le \|\phi\|\|x^*\|_{X^*}.$$

Thus, $\|i^{**}\phi\| \le \|\phi\|$. WE need to show the inverse inequality. Now for each $y^* \in Y^*$, the Hahn-Banach Theorem gives an element $x^* \in X^*$ such that $\|x^*\|_{X^*} = \|y^*\|_{Y^*}$ and $x^*|_Y = y^*$ and hence, $i^*x^* = y^*$. Then we have

$$|\phi(y^*)| = |\phi(x^*|_Y)| = |\phi(i^*x^*)| = |(i^{**}\circ\phi)(x^*)| \le \|i^{**}\phi\|\|x^*\|_{X^*} = \|i^{**}\phi\|\|y^*\|_{Y^*}$$

for all $y^* \in Y^*$. Therefore, we have $\|i^{**}\phi\| = \|\phi\|$.
For Part (ii), let $\psi \in (X/Y)^*$. Note that since $\|\pi^*\| = \|\pi\| \le 1$, we have $\|\pi^*\psi\| \le \|\psi\|$. On the other hand, for each $\bar{x} := \pi(x) \in X/Y$ with $\|\bar{x}\| < 1$, we can choose an element $m \in Y$ such that $\|x + m\| < 1$. Therefore, we have

$$|\psi(\bar{x})| = |\psi\circ\pi(x)| = |\psi\circ\pi(x+m)\| \le \|\psi\circ\pi\| = \|\pi^*(\psi)\|.$$

Therefore, we have $\|\psi\| \le \|\pi^*(\psi)\|$. The proof is complete. □

Remark 4.16. By using Proposition 4.15, we can give an alternative proof of the Riesz's Lemma 2.5.
Using the notations as in Proposition 4.15, if $Y \subsetneq X$, then we have $\|\pi\| = \|\pi^*\| = 1$ because π^* is an isometry by Proposition 4.15(ii). Thus we have $\|\pi\| = \sup\{\|\pi(x)\| : x \in X,\ \|x\| = 1\} = 1$. Hence, for any $0 < \theta < 1$, we can find element $z \in X$ with $\|z\| = 1$ such that $\theta < \|\pi(z)\| = \inf\{\|z + y\| : y \in Y\}$. The Riesz's Lemma follows.

5. Reflexive Spaces

Proposition 5.1. *For a normed space X, let $Q : X \longrightarrow X^{**}$ be the canonical map, that is, $Qx(x^*) := x^*(x)$ for $x^* \in X^*$ and $x \in X$. Then Q is an isometry.*

Proof. Note that for $x \in X$ and $x^* \in B_{X^*}$, we have $|Q(x)(x^*)| = |x^*(x)| \le \|x\|$. Then $\|Q(x)\| \le \|x\|$.
We need to show that $\|x\| \le \|Q(x)\|$ for all $x \in X$. In fact, for $x \in X$, there is $x^* \in X^*$ with $\|x^*\| = 1$ such that $\|x\| = |x^*(x)| = |Q(x)(x^*)|$ by Proposition 4.8. Thus we have $\|x\| \le \|Q(x)\|$. The proof is complete. □

Remark 5.2. Let $T : X \to Y$ be a bounded linear operator and $T^{**} : X^{**} \to Y^{**}$ the second dual operator induced by the adjoint operator of T. Using notations as in Proposition 5.1 above, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle Q_X}\big\downarrow & & \big\downarrow{\scriptstyle Q_Y} \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

Definition 5.3. *A normed space X is said to be reflexive if the canonical map $Q : X \longrightarrow X^{**}$ is surjective. (Note that every reflexive space must be a Banach space.)*

Example 5.4. *We have the following examples.*
- *(i) : Every finite dimensional normed space X is reflexive.*
- *(ii) : ℓ^p is reflexive for $1 < p < \infty$.*
- *(iii) : c_0 and ℓ^1 are not reflexive.*

Proof. For Part (i), if $\dim X < \infty$, then $\dim X = \dim X^{**}$. Hence, the canonical map $Q : X \to X^{**}$ must be surjective.
Part (ii) follows from $(\ell^p)^* = \ell^q$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.
For Part (iii), note that $c_0^{**} = (\ell^1)^* = \ell^\infty$. Since ℓ^∞ is non-separable but c_0 is separable. Therefore, the canonical map Q from c_0 to $c_0^{**} = \ell^\infty$ must not be surjective.
For the case of ℓ^1, we have $(\ell^1)^{**} = (\ell^\infty)^*$. Since ℓ^∞ is non-separable, the dual space $(\ell^\infty)^*$ is non-separable by Proposition 4.11. Therefore, $\ell^1 \ne (\ell^1)^{**}$. □

Proposition 5.5. *Every closed subspace of a reflexive space is reflexive.*

Proof. Let Y be a closed subspace of a reflexive space X. Let $Q_Y : Y \to Y^{**}$ and $Q_X : X \to X^{**}$ be the canonical maps as before. Let $y_0^{**} \in Y^{**}$. We define an element $\phi \in X^{**}$ by $\phi(x^*) := y_0^{**}(x^*|_Y)$ for $x^* \in X^*$. Since X is reflexive, there is $x_0 \in X$ such that $Q_X x_0 = \phi$. Suppose $x_0 \notin Y$. Then by Proposition 4.10, there is $x_0^* \in X^*$ such that $x_0^*(x_0) \ne 0$ but $x_0^*(Y) \equiv 0$. Note that we have $x_0^*(x_0) = Q_X x_0(x_0^*) = \phi(x_0^*) = y_0^{**}(x_0^*|_Y) = 0$. It leads to a contradiction, and so $x_0 \in Y$. The proof is complete if we have $Q_Y(x_0) = y_0^{**}$.
In fact, for each $y^* \in Y^*$, then by the Hahn-Banach Theorem, y^* has a continuous extension x^* in X^*. Then we have

$$Q_Y(x_0)(y^*) = y^*(x_0) = x^*(x_0) = Q_X(x_0)(x^*) = \phi(x^*) = y_0^{**}(x^*|_Y) = y_0^{**}(y^*).$$

□

Example 5.6. *By using Proposition 5.5, we immediately see that the space ℓ^∞ is not reflexive because it contains a non-reflexive closed subspace c_0.*

Proposition 5.7. *Let X be a Banach space. Then we have the following assertions.*
- *(i) X is reflexive if and only if the dual space X^* is reflexive.*
- *(ii) If X is reflexive, then so is every quotient of X.*

Proof. For Part (i), suppose that X is reflexive first. Let $\widetilde{z} \in X^{***}$. Then the restriction $z := \widetilde{z}|_X \in X^*$. Then one can directly check that $Qz = z$ on X^{**} since $X^{**} = X$.
For the converse, assume that X^* is reflexive but X is not. Therefore, X is a proper closed subspace of X^{**}. Then by using the Hahn-Banach Theorem, we can find a non-zero element $\phi \in X^{***}$ such that $\phi(X) \equiv 0$. However, since X^{***} is reflexive, we have $\phi \in X^*$ and hence, $\phi = 0$ which leads to a contradiction.
For Part (ii), we assume that X is reflexive. Let M be a closed subspace of X and $\pi : X \to X/M$ the natural projection. Note that the adjoint operator $\pi^* : (X/M)^* \to X^*$ is an isometry (**Check !**). Thus, $(X/M)^*$ can be viewed as a closed subspace of X^*. By Part (i) and Proposition 5.5, we see that $(X/M)^*$ is reflexive. Then X/M is reflexive by using Part (i) again.
The proof is complete. □

Lemma 5.8. *Let M be a closed subspace of a normed space X. Let $r : X^* \to M^*$ be the restriction map, that is $x^* \in X^* \mapsto x^*|_M \in M^*$. Put $M^\perp := \ker r := \{x^* \in X^* : x^*(M) \equiv 0\}$. Then the canonical linear isomorphism $\widetilde{r} : X^*/M^\perp \to M^*$ induced by r is an isometric isomorphism.*

Proof. We first note that r is surjective by using the Hahn-Banach Theorem. We need to show that $\widetilde{r}$ is an isometry. Note that $\widetilde{r}(x^* + M^\perp) = x^*|_M$ for all $x^* \in X^*$. Now for any $x^* \in X^*$, we have $\|x^* + y^*\|_{X^*} \geq \|x^* + y^*\|_{M^*} = \|x^*|_M\|_{M^*}$ for all $y^* \in M^\perp$. Thus, we have $\|\widetilde{r}(x^* + M^\perp)\| = \|x^*|_M\|_{M^*} \leq \|x^* + M^\perp\|$. We need to show the reverse inequality.
Now for any $x^* \in X^*$, then by the Hahn-Banach Theorem again, there is $z^* \in X^*$ such that $z^*|_M = x^*|_M$ and $\|z^*\| = \|x^*|_M\|_{M^*}$. Then $x^* - z^* \in M^\perp$ and hence, we have $x^* + M^\perp = z^* + M^\perp$. This implies that

$$\|x^* + M^\perp\| = \|z^* + M^\perp\| \leq \|z^*\| = \|x^*|_M\|_{M^*} = \|\widetilde{r}(x^* + M^\perp)\|.$$

The proof is complete. □

Proposition 5.9. (Three-space property): *Let M be a closed subspace of a normed space X. If M and the quotient space X/M both are reflexive, then so is X.*

Proof. Let $\pi : X \to X/M$ be the natural projection. Let $\psi \in X^{**}$. We going to show that $\psi \in im(Q_X)$. Since $\pi^{**}(\psi) \in (X/M)^{**}$, there exists $x_0 \in X$ such that $\pi^{**}(\phi) = Q_{X/M}(x_0 + M)$ because X/M is reflexive. Thus we have

$$\pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0 + M)(\bar{x}^*)$$

for all $\bar{x}^* \in (X/M)^*$. This implies that

$$\psi(\bar{x}^* \circ \pi) = \psi(\pi^*\bar{x}^*) = \pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0 + M)(\bar{x}^*) = \bar{x}^*(x_0 + M) = Q_X x_0(\bar{x}^* \circ \pi)$$

for all $\bar{x}^* \in (X/M)^*$. Therefore, we have

$$\psi = Q_X x_0 \quad \text{on} \quad M^\perp.$$

Therefore, we have $\psi - Q_X(x_0) \in (X^*/M^\perp)^*$. Let $f : M^* \to X^*/M^\perp$ be the inverse of the isometric isomorphism $\widetilde{r}$ which is defined as in Lemma 5.8. Then the composite $(\psi - Q_X x_0) \circ f : M^* \to X^*/M^\perp \to \mathbb{K}$ lies in M^{**}. Then by the reflexivity of M, there is an element $m_0 \in M$ such that

$$(\psi - Q_X x_0) \circ f = Q_M(m_0) \in M^{**}.$$

Notice that for each $x^* \in X^*$, we can find an element $m^* \in M^*$ such that $f(m^*) = x^* + M^\perp \in X^*/M^\perp$ because f is surjective. Moreover, by the construction of $\widetilde{r}$ in Lemma 5.8, we see that $x^*|_M = m^*$. This gives

$$\psi(x^*) - x^*(x_0) = (\psi - Q_X x_0)(m^*) \circ f = Q_M(m_0)(m^*) = m^*(m_0) = x^*(m_0).$$

Thus, we have $\psi(x^*) = x^*(x_0 + m_0)$ for all $x^* \in X^*$. From this we have $\psi = Q_X(x_0 + m_0) \in im(Q_X)$ as desired. The proof is complete. □

Remark 5.10. In view of the definition of a reflexive space, it is naturally raised the question that whether a Banach space X is reflexive whenever it is isometrically isomorphic to its second dual. The answer is negative. A counter example was given by R.C. James in 1951 (see [8]).

6. Weakly convergent and Weak* convergent

Definition 6.1. *Let X be a normed space. A sequence (x_n) is said to be weakly convergent if there is $x \in X$ such that $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, x is called a weak limit of (x_n).*

Proposition 6.2. *A weak limit of a sequence is unique if it exists. In this case, if (x_n) weakly converges to x, denoted by $x = w\text{-}\lim_n x_n$ or $x_n \xrightarrow{w} x$.*

Proof. The uniqueness follows immediately from the Hahn-Banach Theorem. □

Remark 6.3. *Clearly, if a sequence (x_n) converges to $x \in X$ in norm, then $x_n \xrightarrow{w} x$. However, the weakly convergence of a sequence does not imply the norm convergence.*
For example, consider $X = c_0$ and (e_n). Then $f(e_n) \to 0$ for all $f \in c_0^* = \ell^1$ but (e_n) is not convergent in c_0.

Proposition 6.4. *Suppose that X is finite dimensional. A sequence (x_n) in X is norm convergent if and only if it is weakly convergent.*

Proof. Suppose that (x_n) weakly converges to x. Let $\mathcal{B} := \{e_1, .., e_N\}$ be a basis for X and let f_k be the k-th coordinate functional corresponding to the basis $\mathcal{B}$, i.e., $v = \sum_{k=1}^N f_k(v)e_k$ for all $v \in X$. Since $\dim X < \infty$, we have f_k in X^* for all $k = 1, ..., N$. Therefore, we have $\lim_n f_k(x_n) = f_k(x)$ for all $k = 1, ..., N$. Thus, we have $\|x_n - x\| \to 0$. □

Definition 6.5. *Let X be a normed space. A sequence (f_n) in X^* is said to be weak* convergent if there is $f \in X^*$ such that $\lim_n f_n(x) = f(x)$ for all $x \in X$, that is f_n point-wise converges to f. In this case, f is called the weak* limit of (f_n). Write $f = w^*\text{-}\lim_n f_n$ or $f_n \xrightarrow{w^*} f$.*

Remark 6.6. *In the dual space X^* of a normed space X, we always have the following implications:*

$$\text{“Norm Convergent”} \Longrightarrow \text{“Weakly Convergent”} \Longrightarrow \text{“Weak* Convergent”}.$$

However, the converse of each implication does not hold.

Example 6.7. *Remark 6.3 has shown that the w-convergence does not imply $\|\cdot\|$-convergence.*
We now claim that the w^*-convergence also Does Not imply the w-convergence.
Consider $X = c_0$. Then $c_0^* = \ell^1$ and $c_0^{**} = (\ell^1)^* = \ell^\infty$. Let $e_n^* = (0, ...0, 1, 0...) \in \ell^1 = c_0^*$, where the n-th coordinate is 1. Then $e_n^* \xrightarrow{w^*} 0$ but $e_n^* \nrightarrow 0$ weakly because $e^{**}(e_n^*) \equiv 1$ for all n, where $e^{**} := (1, 1, ...) \in \ell^\infty = c_0^{**}$. Hence the w^*-convergence does not imply the w-convergence.

Proposition 6.8. *Let (f_n) be a sequence in X^*. Suppose that X is reflexive. Then $f_n \xrightarrow{w} f$ if and only if $f_n \xrightarrow{w^*} f$.*
In particular, if $\dim X < \infty$, then the followings are equivalent:

- *(i)* : $f_n \xrightarrow{\|\cdot\|} f$;
- *(ii)* : $f_n \xrightarrow{w} f$;
- *(iii)* : $f_n \xrightarrow{w^*} f$.

Theorem 6.9. (**Banach**) : *Let X be a separable normed space. If (f_n) is a bounded sequence in X^*, then it has a w^*-convergent subsequence.*

Proof. Let $D := \{x_1, x_2, ...\}$ be a countable dense subset of X. Note that since $(f_n)_{n=1}^{\infty}$ is bounded, $(f_n(x_1))$ is a bounded sequence in $\mathbb{K}$. Then $(f_n(x_1))$ has a convergent subsequence, say $(f_{1,k}(x_1))_{k=1}^{\infty}$ in $\mathbb{K}$. Let $c_1 := \lim_k f_{1,k}(x_1)$. Now consider the bounded sequence $(f_{1,k}(x_2))$. Then there is convergent subsequence, say $(f_{2,k}(x_2))$, of $(f_{1,k}(x_2))$. Put $c_2 := \lim_k f_{2,k}(x_2)$. Note that we still have $c_1 = \lim_k f_{2,k}(x_1)$. To repeat the same step, if we define $(m,k) \leq (m',k')$ if $m < m'$; or $m = m'$ with $k \leq k'$, we can find a sequence $(f_{m,k})_{m,k}$ in X^* such that

(i) : $(f_{m+1,k})_{k=1}^{\infty}$ is a subsequence of $(f_{m,k})_{k=1}^{\infty}$ for $m = 0, 1, ..,$ where $f_{0,k} := f_k$.
(ii) : $c_i = \lim_k f_{m,k}(x_i)$ exists for all $1 \leq i \leq m$.

Now put $h_k := f_{k,k}$. Then (h_k) is a subsequence of (f_n). Note that for each i, we have $\lim_k h_k(x_i) = \lim_k f_{i,k}(x_i) = c_i$ by the construction (ii) above. Since $(\|h_k\|)$ is bounded and D is dense in X, we have $h(x) := \lim_k h_k(x)$ exists for all $x \in X$ and $h \in X^*$. That is $h = w^*\text{-}\lim_k h_k$. The proof is complete. □

Remark 6.10. *Theorem 6.9 does not hold if the separability of X is removed.*
For example, consider $X = \ell^\infty$ and δ_n the n-th coordinate functional on ℓ^∞. Then $\delta_n \in (\ell^\infty)^*$ with $\|\delta_n\|_{(\ell^\infty)^*} = 1$ for all n. Suppose that (δ_n) has a w^*-convergent subsequence $(\delta_{n_k})_{k=1}^{\infty}$. Define $x \in \ell^\infty$ by

$$x(m) = \begin{cases} 0 & \textit{if } \ m \neq n_k; \\ 1 & \textit{if } \ m = n_{2k}; \\ -1 & \textit{if } m = n_{2k+1}. \end{cases}$$

Hence we have $|\delta_{n_i}(x) - \delta_{n_{i+1}}(x)| = 2$ for all $i = 1, 2, ...$ It leads to a contradiction. Thus(δ_n) has no w^*-convergent subsequence.

Corollary 6.11. *Let X be a separable space. Assume that a sequence in X^* is w^*-convergent if and only if it is norm convergent. Then $\dim X < \infty$.*

Proof. We need to show that the closed unit ball B_{X^*} in X^* is compact in norm. Let (f_n) be a sequence in B_{X^*}. By using Theorem 6.9, (f_n) has a w^*-convergent subsequence (f_{n_k}). Then by the assumption, (f_{n_k}) is norm convergent. Note that if $\lim\limits_k f_{n_k} = f$ in norm, then $f \in B_{X^*}$. Thus B_{X^*} is compact and thus $\dim X^* < \infty$. Thus $\dim X^{**} < \infty$ that gives $\dim X$ is finite because $X \subseteq X^{**}$. □

Corollary 6.12. *Suppose that X is a separable. If X is reflexive space, then the closed unit ball B_X of X is sequentially weakly compact, i.e. it is equivalent to saying that any bounded sequence in X has a weakly convergent subsequence.*

Proof. Let $Q : X \to X^{**}$ be the canonical map as before. Let (x_n) be a bounded sequence in X. Hence, (Qx_n) is a bounded sequence in X^{**}. We first note that since X is reflexive and separable, X^* is also separable by Proposition 4.11. We can apply Theorem 6.9, (Qx_n) has a w^*-convergent subsequence (Qx_{n_K}) in $X^{**} = Q(X)$ and hence, (x_{n_k}) is weakly convergent in X. □

Remark 6.13. In fact, the converse of Corollary 6.12 also holds (see Appendix 7 below). The assumption of separability of X can be removed. We have the following stronger result which was shown by R. C. James (see [10, §1.13]).

Theorem 6.14. *Let X be a Banach space. Then the following are equivalent.*

(i) X is reflexive.
(ii) Every bounded sequence in X has a weakly convergent subsequence.

(iii) The closed unit ball B_X of X is weakly compact, that is, B_X is compact in the weak topology.

7. Appendix: Sequentially weakly compactness and reflexivity for a separable space

This section is devoted to show the theorem that a separable Banach space is reflexive if and only if its closed unit ball is weakly sequentially compact. This results was obtained by Banach [2, Chapter VIII and Chapter XI]. For simply, throughout this section, all Banach spaces are assumed to be over $\mathbb{R}$.

Some set theory

Before showing the main theorem, we need some basic knowledge of the set theory which can be found in the Halmos's classic book [7].
Recall that a partially order set S is called a *well ordered set* if for any non-empty subset A of S contains the least element, that is, there is an element $x_0 \in A$ such that $x_0 \leq x$ for all $x \in A$. In particular, S is automatically a totally order set. The *well-ordering theorem* tells us that every set can be equipped with a well-ordering.
Two well ordered sets A and B are said to have the same *ordinal number or ordinal* for simply if there is an order preserving bijection from A onto B. In particular, each ordinal can be viewed as a well ordered set. More precisely, an ordinal number α is a well ordered set such that for any element $\eta \in \alpha$, we have $\eta = \{\xi \in \alpha : \xi < \eta\}$, thus, we have $\eta \subseteq \alpha$ whenever $\eta \in \alpha$. This definition was due to von Neumann.

Put ω the least infinite countable ordinal, that is, $\omega := \{0, 1, 2...\}$ and is endowed with the usual order.

On the other hand, it is naturally led to define an order on the class of ordinals as the following. (**Warning**: We DO NOT HAVE a statement about " the set of All ordinals numbers" !!!!)

Definition 7.1. *Let α and β be two ordinals. We say that*

(i) $\alpha = \beta$ if there is an order preserving bijection from α onto β.
(ii) $\alpha \leq \beta$ if there is an order preserving injection from α to β.
(iii) $\alpha < \beta$ if $\alpha \leq \beta$ but $\alpha \neq \beta$.

From the von Neumann's definition, we have (see [7, Section 20])

Lemma 7.2. *Every nonempty set of ordinal numbers is a well ordered set.*

Next we need the following definition for comparing the size of two given sets.

Definition 7.3. *Two sets A and B are said to have the same cardinality, write $A \approx B$, if there is a bijection from A onto B.*
The cardinal number of A, write $|A|$, is defined by an ordinal number given by

$$|A| := \min\{\alpha : \ \alpha \text{ is an ordinal number such that } \alpha \approx A\}$$

(Notice that the well ordering theorem and Lemma 7.2 assure the existence of $|A|$.)

Definition 7.4. *We will use the following terminologies later.*

(i) For an ordinal θ, put $\theta^+ := \theta \cup \{\theta\}$ and define $\xi \leq \theta$ for all elements $\xi \in \theta$. Then θ^+ becomes a well ordered set and $\theta < \theta^+$. In this case, θ^+ is called the successor ordinal of θ. Notice that there is no ordinal γ such that $\theta < \gamma < \theta^+$.

(ii) *An ordinal β is called a limit ordinal if there is no ordinal γ such that $\gamma^+ = \beta$. In this case, we are given any ordinal η with $\eta < \beta$, there is an ordinal ξ such that $\eta < \xi < \beta$, write $\xi \to \beta$. For example, ω_1 :=the cardinal of $\mathbb{R}$ and $\omega := \mathbb{N}$ both are limit ordinals.*

(iii) *Let X be a non-empty set. A function $x : [0, \theta) \to X$ is called a* transfinite sequence *in X (or "θ-transfinite sequence") for some ordinal θ, where $[0, \theta)$ denotes the set of all ordinals ξ satisfying $\xi < \theta$. We also write $(x_\xi)_{\xi<\theta}$ for a transfinite sequence. Note that ω-transfinite sequences are the usual sequences defined as before.*
In this case, if θ is a limit ordinal, one can naturally define the similar notation $\lim\limits_{\xi\to\theta} x_\xi$; $\overline{\lim\limits_{\xi\to\theta}} x_\xi$ and $\underline{\lim\limits_{\xi\to\theta}} x_\xi$ in any metric space as in the usual sequences case.
Clearly, one can show that if $(t_\xi)_{\xi<\theta}$ is a bounded θ-sequence in $\mathbb{R}$, then
$$\overline{\lim_{\xi\to\theta}} t_\xi := \lim_{\eta\to\theta} \sup_{\xi\ge\eta} t_\xi = \inf_{\eta<\theta} \sup_{\xi\ge\eta} t_\xi$$
exists.

(iv) *Let θ be a limit ordinal. A subset M of θ is said to be cofinal if for every ordinal $\mu < \theta$, there is $\nu \in M$ such that $\mu < \nu < \theta$. In this case, if $\lim\limits_{\xi\to\theta} x_\xi$ exists, then so does $\lim\limits_{\nu\in M:\nu\to\theta} x_\nu$ and they are the same.*

Lemma 7.5. *Every infinite cardinal is a limit ordinal.*

Proof. If γ is an infinite cardinal and $\gamma = \theta^+$ for some ordinal θ. Then by the definition of the successor ordinal, we have $\theta \approx \theta^+ \approx \gamma$. Hence, $\gamma = \theta^+$ is not a cardinal because $\theta < \theta^+$. □

Lemma 7.6. *Let X be a Banach space. If $(f_\xi)_{\xi<\theta}$ is a norm bounded θ-sequence in X^*, then there is an element $f \in X^*$ such that $\|f\| \le \sup_{\xi<\theta} \|f_\xi\|$ and*

$$f(x) \le \overline{\lim_{\xi\to\theta}} f_\xi(x) \tag{7.1}$$

for all $x \in X$. Consequently, we have

$$\underline{\lim_{\xi\to\theta}} f_\xi(x) \le f(x) \le \overline{\lim_{\xi\to\theta}} f_\xi(x) \tag{7.2}$$

for all $x \in X$.
In this case, f is called a transfinite limit of (f_ξ) (note that f may not be unique).

Proof. Let $M := \sup_{\xi<\theta} \|f_\xi\|$. We first notice that since $(f_\xi)_{\xi<\theta}$ is bounded, $\overline{\lim\limits_{\xi\to\theta}} f_\xi(x)$ exists for all $x \in X$. Hence, one can define a function $p : X \to \mathbb{R}$ by
$$p(x) := \overline{\lim_{\xi\to\theta}} f_\xi(x)$$
for $x \in X$. Clearly, p is a positively homogenous and sub-additive function. We may assume that $p(x_0) > 0$ for some $x_0 \in X$. To see this, if $p(x_0) < 0$, then $p(-x_0) = \overline{\lim} f_\xi(-x_0) \ge \underline{\lim} f_\xi(-x_0) = -\overline{\lim} f_\xi(x_0) > 0$ as desired. Now if we define a linear map f_0 on $\mathbb{R}x_0$ by $f_0(tx_0) := tp(x_0)$, then $f_0(tx_0) \le p(tx_0)$ for all $t \in \mathbb{R}$. Then by the Hahn-Banach Theorem 4.3, there is a linear extension f of f_0 defined on X such that $f(x) \le p(x)$ for all $x \in X$. Notice that since $|f_\xi(x)| \le M\|x\|$ for all $\xi < \theta$ and for all $x \in X$, we have $p(x) \le M\|x\|$. Thus, we have $\|f\| \le M$ as desired. The last assertion is obtained by putting $-x \in X$ into Eq 7.1. The proof is complete. □

Definition 7.7. *A normed subspace Γ of X^* is said to be transfinitely closed if for every norm bounded transfinite θ-sequence $(f_\xi)_{\xi<\theta}$ in Γ for some limit ordinal θ, one can find an element $f \in \Gamma$ satisfying the Eq 7.1 above, that is,*
$$f(x) \le \overline{\lim_{\xi\to\theta}} f_\xi(x)$$

for all $x \in X$.
Clearly, every transfinitely closed subspace is norm closed by considering $\theta = \omega$ in Eq 7.2 above.

Lemma 7.8. *Let Γ be a transfinitely closed subspace of X^* and f_0 be an element in $X^* \setminus \Gamma$. Let $0 < c < dist(f_0, \Gamma)$. Then there is a non-empty finite subset G of X so that there is no element $f \in \Gamma$ satisfying the following:*

$$|f(x) - f_0(x)| \le c\|x\| \quad \textit{for all } x \in G. \tag{7.3}$$

Proof. Let
$$W := \{\gamma : \ \gamma \text{ is a cardinal such that } \gamma \le |X|\}.$$
Now for each element $\gamma \in W$, put $P(\gamma)$ the sentence given by:
whenever G is a subset of X with $|G| = \gamma$, there exists an element $f_\gamma \in \Gamma$ such that $|f_\gamma(x) - f_0(x)| \le c\|x\|$ for all $x \in G$. Let
$$A := \{\gamma \in W : \ P(\gamma) \text{ holds}\}.$$
Clearly, the sentence P holds for the zero cardinal, that is, $0 \in A$. One the other hand, notice that the cardinal $|X| \in W \setminus A$. To see this, we can enumerate the elements in X such that $(x_\xi)_{\xi<|X|}$ because $[0, |X|) = |X|$. Thus, if $P(|X|)$ holds, there is an element $f \in \Gamma$ so that $|f(x) = f_0(x)| \le c\|x\|$ for all $x \in X$. Hence, $\|f - f_0\| \le c$ which contradicts to the choice of c because $c < dist(f_0, \Gamma)$. Therefore, $W \setminus A$ is a non-empty well ordered set by Lemma 7.2 and hence the set $W \setminus A$ contains the least element, say $\mathfrak{m}$.
We will show that if $\mathfrak{m}$ is infinite, then it will lead to a contradiction.
Claim: $P(\mathfrak{m})$ holds if $\mathfrak{m}$ is infinite.
Now let G be any subset of X with $|G| = \mathfrak{m}$, so, we can enumerate the elements in G as a $\mathfrak{m}$-sequence, say $(x_\xi)_{\xi<\mathfrak{m}}$. Now for any ordinal $\eta < \mathfrak{m}$, notice that if we put $a := |(x_\xi)_{\xi<\eta}|$, then $a \le |\eta| \le \eta < \mathfrak{m}$. The minimality of $\mathfrak{m}$ implies that $P(a)$ holds. Hence, there is an element $f_\eta \in \Gamma$ so that
$$|f_\eta(x_\xi) - f_0(x_\xi)| \le c\|x_\xi\| \ \text{ for all } \quad \xi < \eta.$$
Now if $\mathfrak{m}$ is infinite, then $\mathfrak{m}$ is a limit ordinal by Lemma 7.5. Then by the definition of a transfinitely closed set, there is an element $f \in \Gamma$ such that
$$\varliminf_{\eta\to\mathfrak{m}} f_\eta(x) \le f(x) \le \varlimsup_{\eta\to\mathfrak{m}} f_\eta(x)$$
for all $x \in X$. This gives
$$|f(x_\xi) - f_0(x_\xi)| \le c\|x_\xi\| \ \text{ for all } \quad \xi < \mathfrak{m}.$$
Recall that $G = (x_\xi)_{\xi<\mathfrak{m}}$. Hence, $P(\mathfrak{m})$ holds if $\mathfrak{m}$ is infinite. It leads to a contradiction since $\mathfrak{m} \notin A$ by the definition of the set A.
Thus, we can conclude that $\mathfrak{m}$ is finite.
Since $\mathfrak{m} \notin A$, then by the definition of the sentence P we can find a finite subset G of X with $|G| = \mathfrak{m}$ as desired for the Lemma.
You see!! It is a very clever proof, isn't it!!!. ☐

Proposition 7.9. *Let Γ be a transfinitely closed subspace of X^* and f_0 be an element in $X^* \setminus \Gamma$. Then there is an element $x_0 \in X$ such that $f_0(x_0) = 1$ and $f(x_0) = 0$ for all $f \in \Gamma$.*

Proof. Let $0 < c < dist(f_0, \Gamma)$. Recall Lemma 7.8 that there is a non-empty finite subset G of X such that there is no element $f \in \Gamma$ satisfying the following condition:

$$|f(tx) - f_0(tx)| \le c\|tx\| \quad \text{for all } x \in G \text{ and for all } t > 0. \tag{7.4}$$

Notice that we may assume that $\|x\| \le 1$ for all $x \in G$ because G is finite. In particular, by considering $t = 1/k, k = 1, 2..$ into the Eq 7.4 above, we can find a sequence (x_j) in X with $\lim_j x_j = 0$ such that there is no element $f \in \Gamma$ satisfying the condition:

$$|f(x_j) - f_0(x_j)| \le c \quad \text{for all } j = 1, 2..... \tag{7.5}$$

Now let $\widetilde{f_0} := (f_0(x_j)$ and $\widetilde{f} := (f(x_j))$ for $f \in \Gamma$. Then $\widetilde{f_0} \in c_0$ and $F := \{\widetilde{f} : f \in \Gamma\}$ is a subspace of c_0. Note that the Eq 7.5 implies $\|\widetilde{f_0} - \widetilde{f}\|_\infty > c$ for all $f \in \Gamma$. This gives $dist(\widetilde{f_0}, \overline{F}) \ge c > 0$. Then by the Hahn-Banach separation Theorem, there is an element $\varphi = (t_j) \in \ell_1 = c_0^*$ such that $\varphi(\widetilde{f_0}) = 1$ and $\varphi(\widetilde{f}) = 0$ for all $f \in \Gamma$. In particular, we have

$$\sum_{j=1}^{\infty} t_j f_0(x_j) = 1 \quad \text{and} \quad \sum_{j=1}^{\infty} t_j f(x_j) = 0 \text{ for all } f \in \Gamma\ .$$

Therefore, if we put $x_0 := \sum_{j=1}^{\infty} t_j x_j \in X$, then the element x_0 is as desired. □

Recall the notation that for every element x in a Banach space X, $\widehat{x}$ denotes the element in X^{**} given by $x(f) := f(x)$ for all $f \in X^*$. Put $\widehat{A} := \{\widehat{a} : a \in A\} \subseteq X^{**}$ for a subset A of X.

Lemma 7.10. *Let X be a Banach space. Assume that every normed bounded sequence in X has a weakly convergent subsequence. Let D be a countably infinite subset of X. Then every bounded transfinite sequence in $\widehat{D}$ has a transfinite limit in $\widehat{X} := \{\widehat{x} : x \in X\} \subseteq X^{**}$, that is, for every transfinite sequence $(x_\xi)_{\xi<\theta}$ in D, there is an element $z \in X$ such that*

$$f(z) \le \overline{\lim_{\xi\to\theta}} f(x_\xi) \tag{7.6}$$

for all $f \in X^*$.

Proof. Let θ be a limit ordinal and $(x_\xi)_{\xi<\theta}$ be a bounded transfinite sequence in D. By the assumption of D, we can write $D = \{x_i : i = 0, 1, 2...\}$.
Case 1: there is an infinite sequence $(\xi_n)_{n=0}^{\infty}$ in $[0, \theta)$ such that for every ordinal $\eta < \theta$, there is $N \in \mathbb{N}$ such that $\eta < \xi_n < \theta$ for all $n > N$, write $\lim\limits_{n\to\infty} \xi_n = \theta$. In this case, put $x_{\xi_i} := x_i$ for $i = 0, 1, 2....$ Then by assumption, there is a weakly convergent sequence $(x_{\xi_{i_k}})$ with the weak limit, say $z \in X$. This implies that

$$\widehat{z}(f) = f(z) = \lim_{k\to\infty} f(x_{\xi_{i_k}}) \le \overline{\lim_{\xi\to\theta}} f(x_\xi) = \overline{\lim_{\xi\to\theta}} \widehat{x_\xi}(f)$$

for all $f \in X^*$.
The proof is complete if Eq 7.6 also holds for the following case.
Case 2: there is no sequence (ξ_n) in $[0, \theta)$ such that $\lim\limits_{n\to\infty} \xi_n = \theta$.
In this case, for each $x_i \in D$, let $Z_i := \{\xi < \theta : x_\xi = x_i\}$. We will see that Z_i is confinal subset of $[0, \infty)$ for some $i \in \omega$, that is, for any ordinal $\eta < \theta$, there is $\nu \in Z_i$ such that $\eta < \nu < \theta$. To see this, if we assume that every $Z_i, i \in \omega$, is not a cofinal subset of $[0, \theta)$, then for each $i\omega$, there is an ordinal μ_i with $\mu_i < \theta$ such that $\xi \le \mu_i$ for all $\xi \in Z_i$. Now put $\lambda_0 := \mu_0$ and $\lambda_n := \max\{\mu : i = 0, 1, .., n-1\}$. This gives an increasing sequence $(\lambda_n)_{n\in\omega}$ in $[0, \theta)$. Then by the assumption of this case, there is $\lambda \in [0, \theta)$ such that $\lambda_n \le \lambda$ for all $n \in \omega$ and hence, $\xi \le \lambda$ for all $\xi \in [0, \theta)$ because $\bigcup_{i\in\omega} Z_i = [0, \theta)$. This implies that there is no ordinal ξ such thhat $\lambda < \xi < \theta$ that will lead to a contradiction because θ is a limit ordinal.
Now let Z_{i_0} be a confinal subset of $[0, \theta)$ and $z := x_{i_0}$. Then by the definition of Z_{i_0}, $x_\xi = z$ for all $\xi \in Z_{i_0}$. Thus, we have

$$\widehat{z}(f) = f(z) = \lim_{\xi\in Z_{i_0};\xi\to\theta} f(x_\xi) \le \overline{\lim_{\xi\to\theta}} f(x_\xi) = \overline{\lim_{\xi\to\theta}} \widehat{x_\xi}(f).$$

The proof is finished. □

We are now in a position to reach the following main result in this section.

Theorem 7.11. *Let X be a separable Banach space. Then X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence.*

Proof. The necessary condition has been shown in Corollary 6.12.
We are going to show the converse statement. Let D be a countable subset of X.
Claim: The space $\widehat{X} := \{\widehat{x} : x \in X\}$ is transfinitely closed in X^{**}, that is, for every bounded transfinite sequence $(\widehat{x_\xi})_{\xi<\theta}$ in $\widehat{X}$, there is an element $z \in X$ such that

$$f(z) \leq \overline{\lim_{\xi\to\theta}} f(x_\xi)$$

for all $f \in X^*$.
Now for each $n = 1, 2...$ and each x_ξ, $\xi < \theta$, we choose an element $x_\xi^n \in D$ such that

$$\|x_\xi - x_\xi^n\| < \frac{1}{n}.$$

From this, for each $n = 1, 2, ...$, we obtain a bounded a θ-transfinite sequence $(x_\xi^n)_{\xi<\theta}$ in D. For each $n = 1, 2..$, Lemma 7.10 gives an element $z_n \in X$ such that

$$f(z_n) \leq \overline{\lim_{\xi\to\theta}} f(x_\xi^n)$$

for all $f \in X^*$. In addition from this we have $\|z_n\| \leq 1 + \sup_{\xi<\theta} \|x_\xi\|$ for all $n = 1, 2, ...$ Then the necessary condition implies that (z_n) has a weak convergent subsequence (z_{n_j}). Let z be the weak limit of (z_{n_j}). Then we have

$$\begin{aligned} f(z) &= \lim_{j\to\infty} f(z_{n_j}) \\ &\leq \lim_{j\to\infty} \overline{\lim_{\xi\to\theta}} f(x_\xi^{n_j}) \\ &\leq \lim_{j\to\infty} (\overline{\lim_{\xi\to\theta}} f(x_\xi) + \frac{\|f\|}{n_j}) \\ &= \overline{\lim_{\xi\to\theta}} f(x_\xi) \end{aligned}$$

for all $f \in X^*$. The Claim follows.
Finally, if $\widehat{X} \subsetneq X^{**}$, then there is an element $\phi \in X^{**} \setminus \widehat{X}$. Note that $\widehat{X}$ is a closed subspace of X^{**}. Using Lemma 7.9, the transfinite closeness of $\widehat{X}$ implies that there is an element $f_0 \in X^*$ such that $f_0(x) = \widehat{x}(f_0) = 0$ for all $x \in X$ and so, $f_0 = 0$ but $\phi(f_0) = 1$ that is ridiculous. Thus, $\widehat{X} = X^{**}$. The proof is complete. □

8. Appendix: w^*-compactness

Throughout this section X always denotes a normed space. I suppose that the students have learned a standard course of topology before.
Now for each $\varepsilon > 0$ and for finitely many elements $x_1, ..., x_m$ in X, let

$$W(x_1, .., x_m; \varepsilon) := \{f \in X^* : |f(x_i)| < \varepsilon; \forall i = 1, .., m\}.$$

It is noted that $0 \in W(x_1, .., x_m; \varepsilon)$ for any $\varepsilon > 0$ and for all finitely many elements $x_1, ..., x_m$ in X.

Definition 8.1. *The weak*-topology on the dual space X^* is the topology generated by the collection*

$$\{h + W(x_1, .., x_m; \varepsilon) : h \in X^*; \textit{for } \varepsilon > 0 \textit{ and for finitely many } x_1, .., x_m \in X\}.$$

The following result is an important feature for a weak*-continuous linear functional on X^*.

Proposition 8.2. *Let X be a normed space and let $\varphi \in X^{**}$. Then $\varphi : X^* \to \mathbb{K}$ is w^*-continuous if and only if there is a unique element $x_0 \in X$ such that $\varphi(f) = \widehat{x_0}(f) := f(x_0)$ for all $f \in X^*$.*

Proof. The converse is clear. We are going to show the necessary condition.
Assume that φ is w^*-continuous, in particular, φ is w^* continuous at 0. By the definition of w^*-open neighbourhood of 0 given above (Definition 8.1), there are finitely elements $x_1, ..., x_N$ in X and $\delta > 0$ such that $|\varphi(f)| < 1$ whenever $f \in X^*$ satisfies $|f(x_k)| < \delta$ for all $k = 1, ..., N$. Now if we put $L(f) := \max\limits_{k=1,...,N} |f(x_k)|$ for $f \in X^*$, then clearly we have $L(tf) = tL(f)$ for all $f \in X^*$ and $t > 0$. From this, we see that $L(\frac{\delta}{2L(f)} f) < \delta$ while $f \in X^*$ satisfies $L(f) > 0$. Therefore, we have

$$|\varphi(f)| \leq \frac{2}{\delta} L(f) \tag{8.1}$$

whenever $f \in X^*$ with $L(f) > 0$.
Claim 1: Eq 8.1 holds for all $f \in X^*$. To see this, it suffices to show that if $L(f) = 0$, then $\varphi(f) = 0$. In fact, if $L(f) = 0$, then $L(tf) = tL(f) = 0$ for all $t > 0$. In particular, we have $|(tf)(x_k)| = 0 < \delta$ for all $t > 0$ and for all $k = 1, .., N$. This implies that $t|\varphi(f)| = |\varphi(tf)| < 1$ for all $t > 0$. Thus, we have $\varphi(f) = 0$ as desired.
So, from Eq 8.1, we have

$$\bigcap_{k=1}^{N} \ker \widehat{x_k} \subseteq \ker \varphi. \tag{8.2}$$

Claim 2: $\varphi \in span\{\widehat{x_k} : k = 1, ..., N\}$ in X^{**}. To see this, we first notice that if $\widehat{x_j} \in span\{\widehat{x_k} : k \neq j\}$ for some j, then $\bigcap_{k\neq j} \ker \widehat{x_k} \subseteq \ker \widehat{x_j}$ and so, we have $\bigcap_{k=1}^{N} \ker \widehat{x_k} = \bigcap_{k\neq j} \ker \widehat{x_k}$. Therefore, we may assume that $\{\widehat{x_1}, ..., \widehat{x_N}\}$ is a linearly independent set by considering a maximal linearly independent subset of $\{\widehat{x_1}, ..., \widehat{x_N}\}$. In this case, $\{x_1, ..., x_N\}$ is a linearly independent subset of X. Then the Hahn-Banach Theorem implies that there are $x_1^*, ..., x_N^*$ in X^* such that $x_i^*(x_j) = 1$ if $i = j$; otherwise is 0 for $i, j = 1, ..., N$. From this for any $f \in X^*$, we have $\widehat{x_j}(f - \sum_{k=1}^{N} f(x_k)x_k^*) = 0$ for all $j = 1, ..., N$. Eq 8.2 implies that $\varphi(f - \sum_{k=1}^{N} f(x_k)x_k^*) = 0$ for all $f \in X^*$. This gives

$$\varphi(f) = \sum_{k=1}^{N} f(x_k)\varphi(x_k^*) = \sum_{k=1}^{N} \varphi(x_k^*)\widehat{x_k}(f)$$

for all $f \in X^*$. Hence, **Claim 2** follows from

$$\varphi = \sum_{k=1}^{N} \varphi(x_k^*)\widehat{x_k}.$$

Now if we put $x_0 := \sum_{k=1}^{N} \varphi(x_k^*)x_k \in X$, then $\varphi = \widehat{x_0}$. On the other hand, the uniqueness is clear due to the Hahn-Banach Theorem. We finish the proof. ☐

The following is clearly shown by the definition.

Lemma 8.3. *Using the notations as above, we have*

(i) The weak*-topology is Hausdorff.
(ii) Let $f \in X^*$. Then for each open neighborhood V of f, there are $\varepsilon > 0$ and $x_1, .., x_m$ in X such that $f + W(x_1, .., x_m; \varepsilon) \subseteq V$, that is, the collection $\{f + W(x_1, .., x_m; \varepsilon)\}$ forms an open basis at f.

(iii) A sequence (f_n) waek* converges to f in X^* if and only if for each $\varepsilon > 0$ and for finitely many elements $x_1, ..., x_m$ in X, there is a positive integer N such that $f_n - f \in W(x_1, .., x_m; \varepsilon)$ for all $n \geq N$.

Before showing the main result in this section, let us recall that product topologies.
Let $(Z_i)_{i\in I}$ be a collection of topological spaces. Let Z be the usual Cartesian product, that is

$$Z := \prod_{i\in I} Z_i : \{z : I \to \bigcup_{i\in I} Z_i : z(i) \in Z_i; \forall i \in I\}.$$

Let $p_i : Z \to Z_i$ be the natural projection for $i \in I$. The product topology on Z is the weakest topology such that each projection p_i is continuous. More precisely, the following collection forms an open basis for the product topology:

$$\{\bigcap_{i\in J} p_i^{-1}(W_i) : \ J \text{ is a finite subset of } I \text{ and } W_i \text{ is an open subset of } Z_i\}.$$

We have the following famous result in topology.

Theorem 8.4. Tychonoff's Theorem*: The Cartesian product of compact spaces is compact under the product topology.*

The following result is known as the *Alaoglu's Theorem.*

Theorem 8.5. *The closed unit ball B_{X^*} of the dual space X is compact with respect to the weak*-topology.*

Proof. For each $x \in X$, put $Z_x := [-\|x\|, \|x\|] \subseteq \mathbb{R}$. Each Z_x is endowed with the usual subspace topology of $\mathbb{R}$. Then Z_x is a compact set for all $x \in X$. Let

$$Z := \prod_{x\in X} Z_x.$$

Then the set Z is a compact Hausdorff space under the product topology. Define a mapping by

$$T : f \in B_{X^*} \mapsto Tf \in Z; \quad Tf(x) := f(x) \in Z_x \text{ for } x \in X.$$

Then by the definitions of *weak**-topology and the product topology, it is clear that T is a homeomorphism from B_{X^*} onto its image $T(B_{X^*})$. Recall a fact that any closed subset of a compact Hausdorff space is compact. Since Z is compact Haudorsff, it suffices to show that $T(B_{X^*})$ is a closed subset of Z.
Let $z \in \overline{T(B_{X^*})}$. We are going to show that there is an element $f \in B_{X^*}$ such that $f(x) = z(x)$ for all $x \in X$.
Define a function $f : X \to \mathbb{K}$ by

$$f(x) := z(x)$$

for $x \in X$.
Claim : $f(x+y) = f(x)+f(y)$ for all $x, y \in X$. In fact if we fix $x, y \in X$ and for any $\varepsilon > 0$, then by the definition of product topology, there is an element $g \in B_{X^*}$ such that $|g(x+y) - z(x+y)| < \varepsilon$; $|g(x) - z(x)| < \varepsilon$; and $|g(y) - z(y)| < \varepsilon$. Since g is linear, we have $g(x+y) - g(x) - g(y) = 0$. This implies that

$$|z(x+y) - z(x) - z(y)| = |z(x+y) - g(x+y) - (z(x) - g(x)) - (z(y) - g(y))| < 3\varepsilon$$

for all $\varepsilon > 0$. Thus we have $z(x+y) = z(x) = z(y)$. The **Claim** follows.
Similarly, we have $z(\alpha x) = \alpha z(x)$ for all $\alpha \in \mathbb{K}$ and for all $x \in X$.
Therefore, the functional $f(x) := z(x)$ is linear on X. It remains to show f is bounded with $\|f\| \leq 1$. In fact, for any $x \in X$ and any $\varepsilon > 0$, then there is an element $g \in B_{X^*}$ such that

$g(x) - z(x)| < \varepsilon$. Therefore, we have $|f(x)| = |z(x)| \le |g(x)| + \varepsilon \le \|x\| + \varepsilon$. Therefore, f is bounded and $\|f\| \le 1$ as desired. The proof is complete. $\square$

9. Open Mapping Theorem

Let E and F be the metric spaces. A mapping $f : E \to F$ is called an *open mapping* if $f(U)$ is an open subset of F whenever U is an open subset of E.
Clearly, a continuous bijection is a homeomorphism if and only if it is an open map.

Remark 9.1. Warning *An open map need not be a closed map.*
For example, let $p : (x, y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R}$. Then p is an open map but it is not a closed map. In fact, if we let $A = \{(x, 1/x) : x \neq 0\}$, then A is closed but $p(A) = \mathbb{R} \setminus \{0\}$ is not closed.

Lemma 9.2. *Let X and Y be normed spaces and $T : X \to Y$ a linear map. Then T is open if and only if 0 is an interior point of $T(U)$ where U is the open unit ball of X.*

Proof. The necessary condition is obvious.
For the converse, let W be a non-empty subset of X and $a \in W$. Put $b = Ta$. Since W is open, we choose $r > 0$ such that $B_X(a, r) \subseteq W$. Note that $U = \frac{1}{r}(B_X(a, r) - a) \subseteq \frac{1}{r}(W - a)$. Thus, we have $T(U) \subseteq \frac{1}{r}(T(W) - b)$. Then by the assumption, there is $\delta > 0$ such that $B_Y(0, \delta) \subseteq T(U) \subseteq \frac{1}{r}(T(W) - b)$. This implies that $b + rB_Y(0, \delta) \subseteq T(W)$ and so, $T(a) = b$ is an interior point of $T(W)$. $\square$

Corollary 9.3. *Let M be a closed subspace of a normed space X. Then the natural projection $\pi : X \to X/M$ is an open map.*

Proof. Put U and V the open unit balls of X and X/M respectively. Using Lemma 9.2, the result is obtained by showing that $V \subseteq \pi(U)$. Note that if $\bar{x} = \pi(x) \in V$, then by the definition a quotient norm, we can find an element $m \in M$ such that $\|x + m\| < 1$. Hence we have $x + m \in U$ and $\bar{x} = \pi(x + m) \in \pi(U)$. $\square$

Before showing the main result, we have to make use one of important properties of a metric space which is known as the *Baire Category Theorem.* Recall that a subset A of a metric space E is called a *nowhere dense set* if the closure $\overline{A}$ of A has no interior point.

Proposition 9.4. *Let E be a complete metric space with a metric d. If E is a union of a sequence of subsets (A_n) of E, then $int(\overline{A_N}) \neq \emptyset$ for some A_N. Hence, every complete metric space is not a countable union of nowhere dense sets.*

Proof. Let $F_n := \overline{A_n}$. Hence, $E = \bigcup_{n=1}^{\infty} F_n$. Assume that each F_n has no interior points. Fix an element $x_1 \in E$. Let $0 < \eta_1 < 1/2$. Then $B(x_1, \eta_1) \nsubseteq F_1$. Then there is an element $x_2 \in B(x_1, \eta_1) \setminus F_1$. Since F_1 is closed, we can choose $0 < \eta_2 < 1/2^2$ such that $\overline{B(x_2, \eta_2)} \cap F_1 = \emptyset$ and $\overline{B(x_2, \eta_2)} \subseteq \overline{B(x_1, \eta_1)}$. To repeat the same step, we have a sequence of elements (x_k) in E; a decreasing sequence of positive of numbers (η_k) such that for all $k = 1, 2...$ satisfy the following conditions:

(1) $0 < \eta_k < 1/2^k$.
(2) $\overline{B(x_{k+1}, \eta_{k+1})} \subseteq \overline{B(x_k, \eta_k)}$.
(3) $\overline{B(x_{k+1}, \eta_{k+1})} \cap F_k = \emptyset$.

The completeness of E, together with conditions (1) and (2) imply that the sequence (x_k) is a Cauchy sequence and thus, the limit $l := \lim_k x_k$ exists with $l \in \bigcap_{k=1}^{\infty} \overline{B(x_k, \eta_k)}$. Since $E = \bigcup_{n=1}^{\infty} F_n$, the limit $l \in F_K$ for some K. However, it leads to a contradiction because $F_K \cap \overline{B(x_K, \eta_K)} = \emptyset$ by the condition (3) above. $\square$

Lemma 9.5. *Let $T : X \longrightarrow Y$ be a bounded linear surjection from a Banach space X onto a Banach space Y. Then 0 is an interior point of $T(U)$, where U is the open unit ball of X, i.e., $U := \{x \in X : \|x\| < 1\}$.*

Proof. Set $U(r) := \{x \in X : \|x\| < r\}$ for $r > 0$ and so, $U = U(1)$.
Claim 1 : 0 is an interior point of $\overline{T(U(1))}$.
Note that since T is surjective, $Y = \bigcup_{n=1}^{\infty} T(U(n))$. Then by the Baire Category Theorem, there exists N such that *int* $\overline{T(U(N))} \neq \emptyset$. Let y' be an interior point of $\overline{T(U(N))}$. Then there is $\eta > 0$ such that $B_Y(y', \eta) \subseteq \overline{T(U(N))}$. Since $B_Y(y', \eta) \cap T(U(N)) \neq \emptyset$, we may assume that $y' \in T(U(N))$. Let $x' \in U(N)$ such that $T(x') = y'$. Then we have

$$0 \in B_Y(y', \eta) - y' \subseteq \overline{T(U(N))} - T(x') \subseteq \overline{T(U(2N))} = 2N\overline{T(U(1))}.$$

Thus, we have $0 \in \frac{1}{2N}(B_Y(y', \eta) - y') \subseteq \overline{T(U(1))}$. Hence 0 is an interior point of $\overline{T(U(1))}$. The Claim 1 follows.
Therefore there is $r > 0$ such that $B_Y(0, r) \subseteq \overline{T(U(1))}$. This implies that we have

$$B_Y(0, r/2^k) \subseteq \overline{T(U(1/2^k))} \tag{9.1}$$

for all $k = 0, 1, 2....$
Claim 2 : $D := B_Y(0, r) \subseteq T(U(3))$.
Let $y \in D$. By Eq 9.1, there is $x_1 \in U(1)$ such that $\|y - T(x_1)\| < r/2$. Then by using Eq 9.1 again, there is $x_2 \in U(1/2)$ such that $\|y - T(x_1) - T(x_2)\| < r/2^2$. To repeat the same steps, there exists is a sequence (x_k) such that $x_k \in U(1/2^{k-1})$ and

$$\|y - T(x_1) - T(x_2) - ... - T(x_k)\| < r/2^k$$

for all k. On the other hand, since $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 1/2^{k-1}$ and X is Banach, $x := \sum_{k=1}^{\infty} x_k$ exists in X and $\|x\| \leq 2$. This implies that $y = T(x)$ and $\|x\| < 3$.
Thus we the result follows. □

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Theorem 9.6. Open Mapping Theorem : *Using the notations as in Lemma 9.5, then T is an open mapping.*

Proof. The proof is complete by using Lemmas 9.2 and 9.5. □

Proposition 9.7. *Let T be a bounded linear isomorphism between Banach spaces X and Y. Then T^{-1} is bounded.*
Consequently, if $\|\cdot\|$ and $\|\cdot\|'$ both are complete norms on X such that $\|\cdot\| \leq c\|\cdot\|'$ for some $c > 0$, then these two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Proof. The first assertion follows immediately from the Open Mapping Theorem.
Therefore, the last assertion can be obtained by considering the identity map $I : (X, \|\cdot\|) \to (X, \|\cdot\|')$ which is bounded by the assumption. □

Corollary 9.8. *Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. Then the followings are equivalent.*

(i) The image of T is closed in Y.
(ii) There is $c > 0$ such that

$$d(x, \ker T) \leq c\|Tx\|$$

for all $x \in X$.
(iii) If (x_n) is a sequence in X such that $\|x_n + \ker T\| = 1$ for all n, then $\|Tx_n\| \nrightarrow 0$.

Proof. Let Z be the image of T. Then the canonical map $\widetilde{T} : X/kerT \to Z$ induced by T is a bounded linear isomorphism. Note that $\widetilde{T}(\bar{x}) = Tx$ for all $x \in X$, where $\bar{x} := x + \ker T \in X/\ker T$.
For $(i) \Rightarrow (ii)$: suppose that Z is closed. Then Z becomes a Banach space. Then the Open Mapping Theorem implies that the inverse of $\widetilde{T}$ is also bounded. Thus, there is $c > 0$ such that $d(x, \ker T) = \|\bar{x}\|_{X/\ker T} \leq c\|\widetilde{T}(\bar{x})\| = c\|T(x)\|$ for all $x \in X$. The part (ii) follows.
For $(ii) \Rightarrow (i)$, let (x_n) be a sequence in X such that $\lim Tx_n = y \in Y$ exists and so, (Tx_n) is a Cauchy sequence in Y. Then by the assumption, $(\bar{x}_n)$ is a Cauchy sequence in $X/\ker T$. Since $X/\ker T$ is complete, we can find an element $x \in X$ such that $\lim \bar{x}_n = \bar{x}$ in $X/\ker T$. This gives $y = \lim T(x_n) = \lim \widetilde{T}(\bar{x}_n) = \widetilde{T}(\bar{x}) = T(x)$. Therefore, $y \in Z$.
$(ii) \Leftrightarrow (iii)$ is obvious. The proof is complete. □

Proposition 9.9. *Let X and Y be Banach spaces. Let T and K belong to $B(X, Y)$. Suppose that $T(X)$ is closed and K is of finite rank, then the image $(T + K)(X)$ is also closed.*

Proof. Suppose the conclusion does not hold. We write $\bar{z} := z + \ker(T + K)$ for $z \in X$. Then by Corollary 9.8, there is a sequence (x_n) in X such that $\|\bar{x}_n\| = 1$ for all n and $\|(T+K)x_n\| \to 0$. Thus, (x_n) can be chosen so that it is bounded. By passing a subsequence of (x_n) we may assume that $y := \lim_n K(x_n)$ exists in Y because K is of finite rank. Therefore, we have $\lim_n T(x_n) = -y$. Since T has closed range, we have $Tx = -y$ for some $x \in X$. This gives $\lim T(x_n - x) = 0$. Note that the natural map $\widetilde{T}$ is a topological isomorphism from $X/\ker T$ onto $T(X)$ because $T(X)$ is closed. We see that $\|x_n - x + \ker T\| \to 0$ and thus, $\|y - K(x) + K(\ker T)\| = \lim \|K(x_n) - K(x) + K(\ker T)\| = 0$. From this we have $y - Kx = Ku$ for some $u \in \ker T$. In addition, for each n, there is an element $t_n \in \ker T$ so that $\|x_n - x + t_n\| < 1/n$. This implies that

$$\|K(t_n - u)\| \leq \|K(t_n + (x_n - x))\| + \| - K(x_n + x) - K(u)\| \leq \|K\|1/n \to 0.$$

Therefore, we have $\|t_n - u + (\ker T \cap \ker K)\| \to 0$ because $t_n - u \in \ker T$ and the image of $K|\ker T$ is closed. From this we see that $\|t_n - u + \ker(T + K)\| \to 0$.
On the other hand, since $Tx = -y = -Kx - Ku$ and $u \in \ker T$, we have $(T + K)x = -Ku - Tu$ and so, $x + u \in \ker(T + K)$. Then we can now conclude that

$$\|\bar{x}_n\| = \|\bar{x}_n - (\bar{x} + \bar{u})\| \leq \|\bar{x}_n - \bar{x} - \bar{t}_n\| + \|\bar{t}_n - \bar{u}\| \to 0.$$

It contradicts to the choice of x_n such that $\|\bar{x}_n\| = 1$ for all n. The proof is complete. □

Remark 9.10. *In general, the sum of operators of closed ranges may not have a closed range. Before looking for those examples, let us show the following simple useful lemma.*

Lemma 9.11. *Let X be a Banach space. If $T \in B(X)$ with $\|T\| < 1$, then the operator $1 - T$ is invertible, i.e., there is $S \in B(X)$ such that $(1 - T)S = S(1 - T) = 1$.*

Proof. Note that since X is a Banach space, the set of all bounded operators $B(X)$ is a Banach space under the usual operator norm. This implies that the series $\sum_{k=0}^{\infty} T^k$ is convergent in $B(X)$ because $\|T\| < 1$. On the other hand, we have $1 - T^n = (1 - T)(\sum_{k=0}^{n} T^k)$ for all $n = 1, 2....$ Taking $n \to \infty$, we see that $(1 - T)^{-1}$ exists, in fact, $(1 - T)^{-1} = \sum_{k=0}^{\infty} T^k$. □

Example 9.12. Define an operator $T_0 : \ell^\infty \to \ell^\infty$ by

$$T_0(x)(k) := \frac{1}{k} x(k)$$

for $x \in \ell^\infty$ and $k = 1, 2....$ Note that T_0 is injective with $\|T_0\| \leq 1$ and $im\ T_0 \subseteq c_0$. The Open mapping Theorem tells us that the image $im\ T_0$ must not be closed. Otherwise T_0 becomes an isomorphism from ℓ^∞ onto a closed subspace of c_0. It is ridiculous since ℓ^∞ is nonseparable but c_0 is not. Now if we let $T := \frac{1}{2}T_0$, then $\|T\| < 1$ and T is without closed range. Applying Lemma 9.11, we see that the operator $S := 1 - T$ is invertible and thus, S has closed range. Then by our construction $T = 1 - S$ is the sum of two operators of closed ranges but T does not have closed range as required.

10. Closed Graph Theorem

Let $T : X \longrightarrow Y$. The *graph* of T, denoted by $\mathcal{G}(T)$, is defined by the set $\{(x, y) \in X \times Y : y = T(x)\}$.
Now the direct sum $X \oplus Y$ is endowed with the norm $\|\cdot\|_\infty$, i.e., $\|x \oplus y\|_\infty := \max(\|x\|_X, \|y\|_Y)$. We write $X \oplus_\infty Y$ when $X \oplus Y$ is equipped with this norm.
An operator $T : X \longrightarrow Y$ is said to be closed if its graph $\mathcal{G}(T)$ is a closed subset of $X \oplus_\infty Y$, i.e., whenever, a sequence (x_n) of X satisfies the condition $\|(x_n, Tx_n) - (x, y)\|_\infty \to 0$ for some $x \in X$ and $y \in Y$, we have $T(x) = y$.

Theorem 10.1. Closed Graph Theorem : *Let $T : X \longrightarrow Y$ be a linear operator from a Banach space X to a Banach Y. Then T is bounded if and only if T is closed.*

Proof. The part ($\Rightarrow$) is clear.
Assume that T is closed, i.e., the graph $\mathcal{G}(T)$ is $\|\cdot\|_\infty$-closed. Define $\|\cdot\|_0 : X \longrightarrow [0, \infty)$ by

$$\|x\|_0 = \|x\| + \|T(x)\|$$

for $x \in X$. Then $\|\cdot\|_0$ is a norm on X. Let $I : (X, \|\cdot\|_0) \longrightarrow (X, \|\cdot\|)$ be the identity operator. It is clear that I is bounded since $\|\cdot\| \leq \|\cdot\|_0$.
Claim: $(X, \|\cdot\|_0)$ is Banach. In fact, let (x_n) be a Cauchy sequence in $(X, \|\cdot\|_0)$. Then (x_n) and $(T(x_n))$ both are Cauchy sequences in $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$. Since X and Y are Banach spaces, there are $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|T(x_n) - y\|_Y \to 0$. Thus $y = T(x)$ since the graph $\mathcal{G}(T)$ is closed.
Then by Theorem 9.7, the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Hence, there is $c > 0$ such that $\|T(\cdot)\| \leq \|\cdot\|_0 \leq c\|\cdot\|$ and hence, T is bounded since $\|T(\cdot)\| \leq \|\cdot\|_0$. The proof is complete. □

Example 10.2. *Let $D := \{\mathbf{c} = (c_n) \in \ell^2 : \sum_{n=1}^\infty n^2|c_n|^2 < \infty\}$. Define $T : D \longrightarrow \ell^2$ by $T(\mathbf{c}) = (nc_n)$. Then T is an unbounded closed operator.*

Proof. Note that since $\|Te_n\| = n$ for all n, T is not bounded. Now we claim that T is closed.
Let $(\mathbf{x_i})$ be a convergent sequence in D such that $(T\mathbf{x_i})$ is also convergent in ℓ^2. Write $\mathbf{x_i} = (x_{i,n})_{n=1}^\infty$ with $\lim\limits_i \mathbf{x_i} = \mathbf{x} := (x_n)$ in D and $\lim\limits_i T\mathbf{x_i} = \mathbf{y} := (y_n)$ in ℓ^2. This implies that if we fix n_0, then $\lim\limits_i x_{i,n_0} = x_{n_0}$ and $\lim\limits_i n_0 x_{i,n_0} = y_{n_0}$. This gives $n_0 x_{n_0} = y_{n_0}$. Thus $T\mathbf{x} = \mathbf{y}$ and hence T is closed. □

Example 10.3. *Let $X := \{f \in C^b(0,1) \cap C^\infty(0,1) : f' \in C^b(0,1)\}$. Define $T : f \in X \mapsto f' \in C^b(0,1)$. Suppose that X and $C^b(0,1)$ both are equipped with the* sup*-norm. Then T is a closed unbounded operator.*

Proof. Note that if a sequence $f_n \to f$ in X and $f_n' \to g$ in $C^b(0,1)$. Then $f' = g$. Hence T is closed. In fact, if we fix some $0 < c < 1$, then by the Fundamental Theorem of Calculus, we have

$$0 = \lim_n (f_n(x) - f(x)) = \lim_n (\int_c^x (f_n'(t) - f'(t))dt) = \int_c^x (g(t) - f'(t))dt$$

for all $x \in (0,1)$. This implies that we have $\int_c^x g(t)dt = \int_c^x f'(t)dt$. Thus $g = f'$ on $(0,1)$. On the other hand, since $\|Tx^n\|_\infty = n$ for all $n \in \mathbb{N}$. Thus T is unbounded as desired. □

Let X be a normed space and let X^* be its dual space. Then there is a natural bi-linear mapping on $X \times X^*$ (call a dual pair) given by

$$\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{K}; \quad \langle x, f \rangle = f(x).$$

Moreover, this dual pair is non-degenerate, that is, $\langle x, f \rangle = 0$ for all $f \in X^*$ if and only if $x = 0$ and $\langle x, f \rangle = 0$ for all $x \in X$ if and only if $f = 0$.

Proposition 10.4. *Let X and Y be Banach spaces. Let $G : Y^* \to X^*$ be a w^*-w^* continuous linear map. Then we have the following assertions.*

(i) G is bounded.

(ii) There exists a bounded linear map $T \in B(X,Y)$ such that $T^* = G$.

Proof. For showing Part (i), let (y_n^*) be a sequence in Y^* such that $y_n^* \xrightarrow{\|\cdot\|} y^*$ and $Gy_n^* \xrightarrow{\|\cdot\|} x^*$ in the norm topologies. By using the Closed Graph Theorem, we want to show $Gy^* = x^*$, that is, $(Gy^*)(x) = x^*(x)$ for all $x \in X$. In fact, $y_n^* \xrightarrow{\|\cdot\|} y^*$, so $y_n^* \xrightarrow{w^*} y^*$. Thus, we have $Gy_n^* \xrightarrow{w^*} Gy^*$, so $(Gy_n^*)(x) \to (Gy^*)(x)$ for all $x \in X$. On the other hand, since $Gy_n^* \xrightarrow{\|\cdot\|} x^*$, we have $(Gy_n^*)(x) \to x^*(x)$ for all $x \in X$. Therefore, $(Gy^*)(x) = x^*(x)$ for all $x \in X$ as desired.
For Part (ii), note that for each $x \in X$, the map $f \in Y^* \mapsto \langle x, Gf \rangle$ is w^*-continuous on Y^*. Hence, Proposition 8.2 gives a unique element $Rx \in Y$ such that

$$\langle Rx, f \rangle = \langle x, Gf \rangle$$

for all $f \in Y^*$. Then by using Part (i) and the Closed Graph Theorem, R is bounded. The proof is complete. □

11. Uniform Boundedness Theorem

Theorem 11.1. Uniform Boundedness Theorem : *Let $\{T_i : X \longrightarrow Y : i \in I\}$ be a family of bounded linear operators from a Banach space X into a normed space Y. Suppose that for each $x \in X$, we have $\sup\limits_{i \in I} \|T_i(x)\| < \infty$. Then $\sup\limits_{i \in I} \|T_i\| < \infty$.*

Proof. For each $x \in X$, define

$$\|x\|_0 := \max(\|x\|, \sup_{i \in I} \|T_i(x)\|).$$

Then $\|\cdot\|_0$ is a norm on X and $\|\cdot\| \leq \|\cdot\|_0$ on X. If $(X, \|\cdot\|_0)$ is complete, then by the Open Mapping Theorem. This implies that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$ and thus there is $c > 0$ such that

$$\|T_j(x)\| \leq \sup_{i \in I} \|T_i(x)\| \leq \|x\|_0 \leq c\|x\|$$

for all $x \in X$ and for all $j \in I$. Thus$\|T_j\| \leq c$ for all $j \in I$ is as desired.
Thus it remains to show that $(X, \|\cdot\|_0)$ is complete. In fact, if (x_n) is a Cauchy sequence in $(X, \|\cdot\|_0)$, then it is also a Cauchy sequence with respect to the norm $\|\cdot\|$ on X. Write $x := \lim_n x_n$ with respect to the norm $\|\cdot\|$. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|T_i(x_n - x_m)\| < \varepsilon$ for all $m, n \geq N$ and for all $i \in I$. Now fixing $i \in I$ and $n \geq N$ and taking $m \to \infty$, we have $\|T_i(x_n - x)\| \leq \varepsilon$ and thus $\sup_{i \in I} \|T_i(x_n - x)\| \leq \varepsilon$ for all $n \geq N$. Thus, we have $\|x_n - x\|_0 \to 0$ and hence $(X, \|\cdot\|_0)$ is complete. □

Remark 11.2. *Consider* $c_{00} := \{\mathbf{x} = (x_n) : \exists\, N, \forall\, n \geq N; x_n \equiv 0\}$ *which is endowed with* $\|\cdot\|_\infty$. *Now for each* $k \in \mathbb{N}$, *if we define* $T_k \in c_{00}^*$ *by* $T_k((x_n)) := kx_k$, *then* $\sup_k |T_k(\mathbf{x})| < \infty$ *for each* $\mathbf{x} \in c_{00}$ *but* $(\|T_k\|)$ *is not bounded, in fact,* $\|T_k\| = k$. *Thus the assumption of the completeness of* X *in Theorem 11.1 is essential.*

Corollary 11.3. *Let* X *and* Y *be as in Theorem 11.1. Let* $T_k : X \longrightarrow Y$ *be a sequence of bounded operators. Assume that* $\lim_k T_k(x)$ *exists in* Y *for all* $x \in X$. *Then there is* $T \in B(X, Y)$ *such that* $\lim_k \|(T - T_k)x\| = 0$ *for all* $x \in X$. *Moreover, we have* $\|T\| \leq \liminf_k \|T_k\|$.

Proof. Note that by the assumption, we can define a linear operator T from X to Y given by $Tx := \lim_k T_k x$ for $x \in X$. We need to show that T is bounded. In fact, $(\|T_k\|)$ is bounded by the Uniform Boundedness Theorem since $\lim_k T_k x$ exists for all $x \in X$. Hence, for each $x \in B_X$, there is a positive integer K such that $\|Tx\| \leq \|T_K x\| + 1 \leq (\sup_k \|T_k\|) + 1$. Thus, T is bounded.
Finally, it remains to show the last assertion. In fact, note that for any $x \in B_X$ and $\varepsilon > 0$, there is $N(x) \in \mathbb{N}$ such that $\|Tx\| < \|T_k x\| + \varepsilon < \|T_k\| + \varepsilon$ for all $k \geq N(x)$. This gives $\|Tx\| \leq \inf_{k \geq N(x)} \|T_k\| + \varepsilon$ for all $k \geq N(x)$ and hence, $\|Tx\| \leq \inf_{k \geq N(x)} \|T_k\| + \varepsilon \leq \sup_n \inf_{k \geq n} \|T_k\| + \varepsilon$ for all $x \in B_X$ and $\varepsilon > 0$. Therefore, we have $\|T\| \leq \liminf_k \|T_k\|$. □

Corollary 11.4. *Every weakly convergent sequence in a normed space must be bounded.*

Proof. Let (x_n) be a weakly convergent sequence in a normed space X. If we let $Q : X \to X^{**}$ be the canonical isometry, then (Qx_n) is a bounded sequence in X^{**}. Note that (x_n) is weakly convergent if and only if (Qx_n) is w^*-convergent. Thus, $(Qx_n(x^*))$ is bounded for all $x^* \in X^*$. Note that the dual space X^* must be complete. We can apply the Uniform Boundedness Theorem to see that (Qx_n) is bounded and so is (x_n). □

12. Projections on Banach Spaces

Throughout this section, let X be a Banach space. A linear operator $P : X \to X$ is called a *projection (or idempotent)* if it is bounded and satisfies the condition $P^2 = P$.
In addition, a closed subspace E of X is said to be complemented if there is a closed subspace F of X such that $X = E \oplus F$.

Proposition 12.1. *A closed subspace* E *of* X *is complemented if and only if there is a projection* Q *on* X *with* $E = im\ Q$.

Proof. We first suppose that there is a closed subspace F of X such that $X = E \oplus F$. Define an operator $Q : X \to X$ by $Qx = u$ if $x = u + v$ for $u \in E$ and $v \in F$. It is clear that we have $Q^2 = Q$. For showing the boundedness of Q, by using the Closed Graph Theorem, we need to show that if (x_n) is a sequence in E such that $\lim x_n = x$ and $\lim Qx_n = u$ for some $x, u \in E$, then $Qx = u$.
Indeed, if we let $x_n = y_n + z_n$ where $y_n \in E$ and $z_n \in F$, then $Qx_n = y_n$. Note that (z_n) is a convergent sequence in F because $z_n = x_n - y_n$ and (x_n) and (y_n) both are convergent. Let $w = \lim z_n$. This implies that

$$x = \lim x_n = \lim(y_n + z_n) = u + w.$$

Since E and F are closed, we have $u \in E$ and $w \in F$. Therefore, we have $Qx = u$ as desired.
The converse is clear. In fact, we have $X = im\ Q \oplus \ker Q$ in this case. □

Example 12.2. *If* M *is a finite dimensional subspace of a normed space* X, *then* M *is complemented in* X.
In fact, if M *is spanned by* $\{e_i : i = 1, 2.., m\}$, *then* M *is closed and by the Hahn-Banach Theorem,*

for each $i = 1, ..., m$, there is $e_i^* \in X^*$ such that $e_i^*(e_j) = 1$ if $i = j$, otherwise, it is equal to 0. Put $N := \bigcap_{i=1}^m \ker e_i^*$. Then $X = M \oplus N$.

The following example can be found in [4].

Example 12.3. *c_0 is not complemented in ℓ^∞.*

Proof. It will be shown by the contradiction. Suppose that c_0 is complemented in ℓ^∞.
Claim 1: There is a sequence (f_n) in $(\ell^\infty)^*$ such that $c_0 = \bigcap_{n=1}^\infty \ker f_n$.
In fact, by the assumption, there is a closed subspace F of ℓ^∞ such that $\ell^\infty = c_0 \oplus F$. If we let P be the projection from ℓ^∞ onto F along this decomposition, then $\ker P = c_0$ and P is bounded by the Closed Graph Theorem. Let $e_n^* : \ell^\infty \to \mathbb{K}$ be the n-th coordinate functional. Then $e_n^* \in (\ell^\infty)^*$. Thus, if we put $f_n = e_n^* \circ P$, then $f_n \in (\ell^\infty)^*$ and $c_0 = \bigcap_{n=1}^\infty \ker f_n$ as desired.
Claim 2: For each irrational number $\alpha \in [0, 1]$, there is an infinite subset N_α of $\mathbb{N}$ such that $N_\alpha \cap N_\beta$ is a finite set if α and β both are distinct irrational numbers in $[0, 1]$.
In fact, we write $[0, 1] \cap \mathbb{Q}$ as a sequence (r_n). Then for each irrational α in $[0, 1]$, there is a subsequence (r_{n_k}) of (r_n) such that $\lim_k r_{n_k} = \alpha$. Let $N_\alpha := \{n_k : k = 1, 2...\}$. From this, we see that $N_\alpha \cap N_\beta$ is a finite set whenever $\alpha, \beta \in [0, 1] \cap \mathbb{Q}^c$ with $\alpha \neq \beta$. Claim 2 follows.
Now for each $\alpha \in [0, 1] \cap \mathbb{Q}^c$, define an element $x_\alpha \in \ell^\infty$ by $x_\alpha(k) \equiv 1$ as $k \in N_\alpha$; otherwise, $x_\alpha(k) \equiv 0$.
Claim 3: If $f \in (\ell^\infty)^*$ with $c_0 \subseteq \ker f$, then for any $\eta > 0$, the set $\{\alpha \in [0, 1] \cap \mathbb{Q}^c : |f(x_\alpha)| \geq \eta\}$ is finite.
Note that by considering the decomposition $f = Re(f) + iIm(f)$, it suffices to show that the set $\{\alpha \in [0, 1] \cap \mathbb{Q}^c : f(x_\alpha) \geq \eta\}$ is finite. Let $\alpha_1, ...\alpha_N$ in $[0, 1] \cap \mathbb{Q}^c$ such that $f(x_{\alpha_j}) \geq \eta$, $j = 1, ..., N$. Now for each $j = 1, .., N$, set $y_j(k) \equiv 1$ as $k \in N_{\alpha_j} \setminus \bigcup_{m \neq j} N_{\alpha_m}$; otherwise $y_j \equiv 0$. Note that $x_{\alpha_j} - y_j \in c_0$ since $N_\alpha \cap N_\beta$ is finite for $\alpha \neq \beta$ by Claim 2. Hence, we have $f(x_{\alpha_j}) = f(y_j)$ for all $j = 1, ..., N$. Moreover, we have $\{k : y_j(k) = 1\} \cap \{k; y_i(k) = 1\} = \emptyset$ for $i, j = 1, ..., N$ with $i \neq j$. Thus, we have $\|y\|_\infty = 1$. Now we can conclude that

$$\|f\| \geq f(\sum_{j=1}^N y_j) = \sum_{j=1}^N f(x_{\alpha_j}) \geq N\eta.$$

This implies that $|\{\alpha : f(\alpha) \geq \eta\}| \leq \|f\|/\eta$. Claim 3 follows.
We are now going to complete the proof. Now let (f_n) be the sequence in $(\ell^\infty)^*$ as found in the Claim 1. Claim 3 implies that the set $S := \bigcup_{n=1}^\infty \{\alpha \in \mathbb{Q}^c \cap [0, 1] : f_n(x_\alpha) \neq 0\}$ is countable. Thus, there exists $\gamma \in [0, 1] \cap \mathbb{Q}^c$ such that $\gamma \notin S$. Thus, we have $x_\gamma \in \bigcap_{n=1}^\infty \ker f_n$. Besides, since N_γ is an infinite set, we see that $x_\gamma \notin c_0$. Therefore, we have $c_0 \subsetneq \bigcap \ker f_k$ which contradicts to Claim 1. □

Proposition 12.4. (Dixmier) *Let X be a normed space. Let $i : X \to X^{**}$ and $j : X^* \to X^{***}$ be the natural embeddings. Then the composition $Q := j \circ i^* : X^{***} \to X^{***}$ is a projection with $Q(X^{***}) = X^*$.*
Consequently, X^* is a complemented closed subspace of X^{***}.

Proof. Clearly, Q is bounded. Note that $i^* \circ j = Id_{X^*} : X^* \to X^*$. From this, we see that $Q^2 = Q$ as desired.
We need to show that *im* $Q = X^*$, more precisely, *im* $Q = j(X^*)$. In fact, it follows from $Q \circ j = j$ by using the equality $i^* \circ j = Id_{X^*}$ again.
The last assertion follows immediately from Proposition 12.1. □

Corollary 12.5. *c_0 is not isomorphic to the dual space of a normed space.*

Proof. Suppose not. Let $T : c_0 \to X^*$ be an isomorphism from c_0 onto the dual space of some normed space X. Then $T^{**} : c_0^{**} = \ell^\infty \to X^{***}$ is an isomorphism too. Let $Q : X^{***} \to X^{***}$ be the projection with $im\ Q = X^*$ which is found in Proposition 12.4.
Now put $P := (T^{**})^{-1} \circ Q \circ T^{**} : \ell^\infty \to \ell^\infty$. Then P is a projection.
On the other hand, we always have $T^{**}|_{c_0} = T$ (see Remark 5.2). This implies that $im\ P = c_0$. Thus, c_0 is complemented in ℓ^∞ by Proposition 12.1 which leads to a contradiction by Example 12.3. □

Recall that a closed subspace M of a Banach space E is called an *M-ideal* if the space $M^\perp := \{x^* \in E^* : x^*(M) \equiv 0\}$ is a ℓ_1-direct summand of E^*, that is, there is another closed subspace N of E^* such that $E^* = M^\perp \bigoplus_{\ell_1} N$, i.e., for every element $x^* \in E^*$ satisfies the condition: $x^* = u + v$ and $\|x^*\| = \|u\| + \|v\|$ for a pair of elements u and v in $M^\perp$ and N respectively.

Proposition 12.6. *We keep the notation as give in Proposition 12.4. If X is viewed as a closed subspace of X^{**} and suppose that $X^{***} = X^\perp \bigoplus_{\ell_1} N$ for some closed subspace N of X^{***}, i.e. X is an M-ideal of X^{**}, then $N = X^*$.*

Proof. Let $Q : X^{***} \to X^{***}$ be the projection given in Proposition 12.4. Recall that $Qz = j(z|_X)$ for $z \in X^{***}$ and $im\ Q = X^*$. Moreover $\|Q\| \le 1$. Note that $\ker Q = X^\perp := \{z \in X^{***} : z|_X \equiv 0\}$ and hence, $X^{***} = X^\perp \bigoplus X^*$. Let $z \in N$. Then we have $Q(z) = (Q(z) - z) + z \in X^\perp \bigoplus_{\ell_1} N$ and hence, $\|Q(z)\| = \|Q(z) - z\| + \|z\|$. Since $\|Q\| \le 1$, we see that $\|Q(z) - z\| = 0$ and thus, $z = Q(z) \in X^*$. Therefore, we have $N \subseteq X^*$, so $N = X^*$. The proof is complete. □

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Proposition 12.7. *The c_0 space is an M-ideal of ℓ_∞.*

Proof. We first notice that for $h \in (\ell_\infty)^*$ and $\xi \in \ell_\infty$, then $\mathcal{R}e(h)(\xi) := \mathcal{R}e(h(\xi))$ can be viewed as a $\mathbb{R}$-linear functional on ℓ_∞ and $\|h\| = \|\mathcal{R}e(h)\|$.
Using Proposition 12.6, it suffices to show that for $g \in c_0^* = \ell_1$ and $f \in c_0^\perp$, we have $\|g + f\|_{(\ell_\infty)^*} = \|g\|_{(\ell_\infty)^*} + \|f\|_{(\ell_\infty)^*}$, where $c_0^\perp := \{f \in (\ell_\infty)^* : f(c_0) \equiv 0\}$. Let $\varepsilon > 0$. By considering the polar decomposition, then there are elements ξ and ξ' in $(\ell_\infty)_1$ of norm-one such that

$$\|f\| - \varepsilon < f(\xi) \quad \text{and} \quad \|g\| - \varepsilon < g(\xi') = \mathcal{R}e(g)(\xi') = \sum_{n=1}^{\infty} \mathcal{R}e(\xi'(n)g(n)).$$

Since $g \in c_0^* = \ell_1$, there is N such that $\sum_{n>N} |g(n)| < \varepsilon$. Now let ξ'' be an element in ℓ_∞ given by

$$\xi''(n) := \begin{cases} \xi'(n) & \text{if } n \le N \\ \xi(n) & \text{if } n > N. \end{cases}$$

Then $\|\xi''\|_\infty \le 1$ and $\xi'' - \xi \in c_{00}$. Hence we have $f(\xi) = f(\xi'')$ because $f(c_0) \equiv 0$.
On the other hand, since $\sum_{n>N} |g(n)| < \varepsilon$, we have

$$\|g\| - \varepsilon < \mathcal{R}e(g)(\xi') \le \sum_{n=1}^{N} \mathcal{R}e(\xi'(n)g(n)) + \sum_{n>N} |g(n)| < \sum_{n=1}^{N} \mathcal{R}e(\xi'(n)g(n)) + \varepsilon.$$

Thus, we have

$$\begin{aligned}\mathcal{R}e(g)(\xi'') &= \sum_{n=1}^{\infty} \mathcal{R}e(\xi''(n)g(n)) \\ &\geq \sum_{n=1}^{N} \mathcal{R}e(\xi'(n)g(n)) - |\sum_{n>N} \xi(n)g(n)| \\ &\geq \|g\| - 2\varepsilon - \varepsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|g\| + \|f\| &= \|\mathcal{R}e(g)\| + \|f\| \\ &\leq \mathcal{R}e(g)(\xi'') + f(\xi'') + 4\varepsilon \\ &= \mathcal{R}e(g+f)(\xi'') + 4\varepsilon \\ &\leq \|\mathcal{R}e(g+f)\| + 4\varepsilon \\ &= \|g+f\| + 4\varepsilon.\end{aligned}$$

for all $\varepsilon > 0$. The proof is complete.

□

13. Appendix: Basic sequences

Throughout this section, X always denotes a Banach space.
An infinite sequence (x_n) in X is called a *basic sequence* if for each element x in $X_0 := [x_1, x_2, \cdots]$, the closed linear span of $\{x_1, x_2, ...\}$, then there is a unique sequence of scalars (a_n) such that $x = \sum_{i=1}^{\infty} a_i x_i$. Put ψ_i the corresponding i-th coordinate function, i.e., $\psi(x) := a_i$ and $Q_n : X_0 \to E_n := [x_1 \cdots x_n]$ the n-th canonical projection, i.e., $Q_n(\sum_{i=1}^{\infty} a_i x_i) := \sum_{i=1}^{n} a_i x_i$.

Theorem 13.1. *Using the notations as above, for each element $x \in X_0$, put*

$$q(x) := \sup\{\|Q_n(x)\| : n = 1, 2...\}.$$

Then

(i) q is a Banach equivalent norm on X_0.
(ii) Each coordinate projection Q_n and coordinate function ψ_n are bounded in the original norm-topology.

Proof. Since $x = \lim_n Q_n x$ for all $x \in X_0$, we see that q is a norm on X_0 and $q(\cdot) \geq \frac{1}{2}\|\cdot\|$ on X_0. From this, together with the Open Mapping Theorem, all assertions follows if we show that q is a Banach norm on X_0.
Let (x_n) be a Cauchy sequence in X_0 with respect to the norm q. Clearly, (x_n) is also a Cauchy sequence in the $\|\cdot\|$-topology because $q(\cdot) \geq \frac{1}{2}\|\cdot\|$. Let $x = \lim_n x_n$ be the limit in X_0 in the $\|\cdot\|$-topology. We are going to show that x is also the limit of (x_n) with respect to the q-topology. We first note that $y_k := \lim_n Q_k x_n$ exists in X_0 for all $k = 1, 2, ...$ by the definition of the norm q.
Claim 1: $\|\cdot\|$-$\lim_k y_k = x$.
Let $\varepsilon > 0$. Then by the definition of the norm q, there is a positive integer N_1 such that $\|Q_k x_N - Q_k x_m\| < \varepsilon$ and $\|x_N - x_m\| < \varepsilon$ for all $m, N \geq N_1$ and for all $k = 1, 2....$ This gives

$$\|x - Q_k x_m\| \leq \|x - x_{N_1}\| + \|x_{N_1} - Q_k x_{N_1}\| + \|Q_k x_{N_1} - Q_k x_m\| < 2\varepsilon + \|x_{N_1} - Q_k x_{N_1}\|$$

for all $m \geq N_1$ and for all positive integers k. Thus, if we take $m \to \infty$, then we have

$$\|x - y_k\| \leq 2\varepsilon + \|x_{N_1} - Q_k x_{N_1}\| \to 2\varepsilon + 0 \quad \text{as } k \to \infty.$$

Claim 2: $Q_k x = y_k$ for all $k = 1, 2....$
Fix a positive integer k_1. Note that $Q_{k_1} y_k = y_{k_1}$ for all $k \geq k_1$. Indeed, since E_k and E_{k_1} are of

finite dimension, the restrictions $Q_{k_1}|E_k$ and $Q_k|E_{k_1}$ both are continuous. This implies that

$$Q_{k_1}y_k = Q_{k_1}(\lim_n Q_k x_n) = \lim_n Q_{k_1}Q_k(x_n) = \lim_n Q_k Q_{k_1}(x_n) = Q_k(\lim_n Q_{k_1}x_n) = Q_k(y_{k_1}) = y_{k_1}$$

for all $k \geq k_1$. Henece, there is a sequence of scalars (β_n) so that $y_k = \sum_{i=1}^{k} \beta_i x_i$ for all $k = 1, 2...$ On the other hand, if we let $x = \sum_{i=1}^{\infty} \alpha_i x_i$, then by Claim 1 we have $\lim_k(y_k - Q_k x) = 0$ and thus we have $\sum_{i=1}^{\infty}(\beta_i - \alpha_i)x_i = 0$. Therefore, we have $\beta_i = \alpha_i$ for all $i = 1, 2....$ The Claim 2 follows. It remains to show that $\lim_n q(x_n - x) = 0$.

Let $\eta > 0$. Then there is a positive integer N so that $\|Q_k x_n - Q_k x_m\| < \eta$ for all $m, n \geq N$ and for all positive integers k. Taking $m \to \infty$, Claim 2 gives

$$\|Q_k x_n - Q_k x\| = \|Q_k x_n - y_k\| \leq \eta$$

for all $n \geq N$ and for all positive integers k. Thus, we have $q(x_n - x) \leq \eta$ for all $n \geq N$. The proof is complete. □

14. Geometry of Hilbert space I

From now on, all vectors spaces are over the complex field. Recall that an *inner product* on a vector space V is a function $(\cdot,\cdot) : V \times V \to \mathbb{C}$ which satisfies the following conditions.

(i) $(x,x) \geq 0$ for all $x \in V$ and $(x,x) = 0$ if and only if $x = 0$.
(ii) $\overline{(x,y)} = (y,x)$ for all $x, y \in V$.
(iii) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.

Consequently, for each $x \in V$, the map $y \in V \mapsto (x,y) \in \mathbb{C}$ is conjugate linear by the conditions *(ii)* and *(iii)*, i.e., $(x, \alpha y + \beta z) = \bar{\alpha}(x,y) + \bar{\beta}(x,z)$ for all $y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.
In addition, the inner product $(\cdot,\cdot)$ will give a norm on V which is defined by

$$\|x\| := \sqrt{(x,x)}$$

for $x \in V$.

We first recall the following useful properties of an inner product space which can be found in the standard text books of linear algebras.

Proposition 14.1. *Let V be an inner product space. For all $x, y \in V$, we always have:*

(i): (**Cauchy-Schwarz inequality**)*: $|(x,y)| \leq \|x\|\|y\|$ Consequently, the inner product on $V \times V$ is jointly continuous.*

(ii): (**Parallelogram law**)*: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.*
Furthermore, a norm $\|\cdot\|$ on a vector space X is induced by an inner product if and only if it satisfies the Parallelogram law. In this case such inner product is given by the following:

$$\mathfrak{Re}(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \quad \textit{and} \quad \mathfrak{Im}(x,y) = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$$

for all $x, y \in X$.

(iii) **Gram-Schmidt process** *Let $\{x_1, x_2, ...\}$ be a sequence of linearly independent vectors in an inner product space V. Put $e_1 := x_1/\|x_1\|$. Define e_n inductively on n by*

$$e_{n+1} := \frac{x_n - \sum_{k=1}^{n}(x_k, e_k)e_k}{\|x_n - \sum_{k=1}^{n}(x_k, e_k)e_k\|}.$$

Then $\{e_n : n = 1, 2, ..\}$ forms an orthonormal system in V Moreover, the linear span of $x_1, ..., x_n$ is equal to the linear span of $e_1, ..., e_n$ for all $n = 1, 2....$

Clearly, due to the Cauchy-Schwarz inequality, we have the following.

Lemma 14.2. *If X is an inner product space, then the inner product $(\cdot,\cdot) : X \times X \to \mathbb{C}$ is jointly continuous, that is $\lim_n (x_n, y_n) = (x,y)$ whenever (x_n) and (y_n) are convergent sequences in X with $\lim_n x_n = x$ and $\lim_n y_n = y$.*

The following is one of the most important classes in mathematics.

Definition 14.3. *A Hilbert space is a Banach space whose norm is given by an inner product.*

Example 14.4. *(1) $\mathbb{C}^N$ is a Hilbert space under the usual inner product given by $(w,z) := \sum_{k=1}^{N} w(k)\overline{z(k)}$ for $w, z \in \mathbb{C}^N$.*

(2) It follows from Proposition 14.1 immediately that ℓ^2 is a Hilbert space and ℓ^p is not a Hilbert space for all $p \in [1,\infty] \setminus \{2\}$.

In the rest of this section, X always denotes a complex Hilbert space with an inner product $(\cdot,\cdot)$. Recall that two vectors x and y in an inner product space V are said to be *orthogonal* if $(x,y)=0$.

Proposition 14.5. (**Bessel's inequality**) : *Let $\{e_1,...,e_N\}$ be an orthonormal set in an inner product space V, i.e., $(e_i,e_j)=1$ if $i=j$, otherwise is equal to 0. Then for any $x\in V$, we have*

$$\sum_{i=1}^{N}|(x,e_i)|^2\leq\|x\|^2.$$

Proof. It can be obtained by the following equality immediately

$$\|x-\sum_{i=1}^{N}(x,e_i)e_i\|^2=\|x\|^2-\sum_{i=1}^{N}|(x,e_i)|^2.$$

□

Corollary 14.6. *Let $(e_i)_{i\in I}$ be an orthonormal set in a Hilbert space X. Then for any element $x\in X$, the set*

$$I(x):=\{i\in I:(e_i,x)\neq 0\}$$

is countable.

Proof. Note that for each $x\in V$, we have

$$\{i\in I:(e_i,x)\neq 0\}=\bigcup_{n=1}^{\infty}\{i\in I:|(e_i,x)|\geq 1/n\}.$$

Then the Bessel's inequality implies that the set $\{i\in I:|(e_i,x)|\geq 1/n\}$ must be finite for each $n\geq 1$. Thus the result follows. □

Proposition 14.7. *Let (e_n) be a sequence of orthonormal vectors in a Hilbert space X. Then for any $x\in V$, the series $\sum_{n=1}^{\infty}(x,e_n)e_n$ is convergent.*
Moreover, if $(e_{\sigma(n)})$ is a rearrangement of (e_n), i.e., $\sigma:\{1,2...\}\longrightarrow\{1,2,..\}$ is a bijection. Then we have

$$\sum_{n=1}^{\infty}(x,e_n)e_n=\sum_{n=1}^{\infty}(x,e_{\sigma(n)})e_{\sigma(n)}.$$

Proof. Since X is a Hilbert space, the convergence of the series $\sum_{n=1}^{\infty}(x,e_n)e_n$ follows from the Bessel's inequality. In fact, if we put $s_p:=\sum_{n=1}^{p}(x,e_n)e_n$, then we have

$$\|s_{p+k}-s_p\|^2=\sum_{p+1\leq n\leq p+k}|(x,e_n)|^2.$$

Now put $y=\sum_{n=1}^{\infty}(x,e_n)e_n$ and $z=\sum_{n=1}^{\infty}(x,e_{\sigma(n)})e_{\sigma(n)}$. Note that we have

$$\begin{aligned}(y,y-z)&=\lim_N(\sum_{n=1}^{N}(x,e_n)e_n,\sum_{n=1}^{N}(x,e_n)e_n-z)\\&=\lim_N\sum_{n=1}^{N}|(x,e_n)|^2-\lim_N\sum_{n=1}^{N}(x,e_n)\sum_{j=1}^{\infty}\overline{(x,e_{\sigma(j)})}(e_n,e_{\sigma(j)})\\&=\sum_{n=1}^{\infty}|(x,e_n)|^2-\lim_N\sum_{n=1}^{N}(x,e_n)\overline{(x,e_n)}\ (\because\text{ for each }n,\text{ there is a unique }j\text{ such that }n=\sigma(j))\\&=0.\end{aligned}$$

Similarly, we have $(z,y-z)=0$. The result follows. □

A family of an orthonormal vectors, say $\mathcal{B}$, in X is said to be **complete** if it is maximal with respect to the set inclusion order, i.e., if $\mathcal{C}$ is another family of orthonormal vectors with $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.
A complete orthonormal subset of X is also called an **orthonormal basis** of X.

Proposition 14.8. *Let $\{e_i\}_{i\in I}$ be a family of orthonormal vectors in X. Then the followings are equivalent:*

(i): $\{e_i\}_{i\in I}$ is complete;
(ii): if $(x, e_i) = 0$ for all $i \in I$, then $x = 0$;
(iii): for any $x \in X$, we have $x = \sum_{i\in I}(x, e_i)e_i$;
(iv): for any $x \in X$, we have $\|x\|^2 = \sum_{i\in I} |(x, e_i)|^2$.

Note : *there are only countable many $(x, e_i) \neq 0$ by Corollary 14.6, so the sums in (iii) and (iv) are convergent by Proposition 14.7. In this case, the expression of each element $x \in X$ in Part (iii) is unique.*

Proposition 14.9. *Let X be a Hilbert space. Then*

(i) : X processes an orthonormal basis.
(ii) : If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ both are the orthonormal bases for X, then I and J have the same cardinality. In this case, the cardinality $|I|$ of I is called the orthonormal dimension of X.

Proof. Part (i) follows from Zorn's Lemma.
For part (ii), if the cardinality $|I|$ is finite, then the assertion is clear since $|I| = \dim X$ (*vector space dimension*) in this case.
Now assume that $|I|$ is infinite, for each e_i, put $J_{e_i} := \{j \in J : (e_i, f_j) \neq 0\}$. Note that since $\{e_i\}_{i\in I}$ is maximal, Proposition 14.8 implies that we have

$$J = \bigcup_{i\in I} J_{e_i}.$$

Note that J_{e_i} is countable for each e_i by using Proposition 14.6. On the other hand, we have $|\mathbb{N}| \leq |I|$ because $|I|$ is infinite and thus $|\mathbb{N} \times I| = |I|$. Then we have

$$|J| \leq \sum_{i\in I} |J_{e_i}| = \sum_{i\in I} |\mathbb{N}| = |\mathbb{N} \times I| = |I|.$$

Similarly, we also have $|I| \leq |J|$. □

Remark 14.10. *Recall that a vector space dimension of X is defined by the cardinality of a maximal linearly independent set in X.*
Note that if X is finite dimensional, then the orthonormal dimension is the same as the vector space dimension.
In addition, the vector space dimension is larger than the orthornormal dimension in general since every orthogonal set must be linearly independent.

Two Hilbert spaces X and Y are said to be *isomorphic* if there is linear isomorphism U from X onto Y such that $(Ux, Ux') = (x, x')$ for all $x, x' \in X$. In this case U is called a *unitary operator.*

Theorem 14.11. *Two Hilbert spaces are isomorphic if and only if they have the same orthonornmal dimension.*

Proof. The converse part ($\Leftarrow$) is clear.
Now for the ($\Rightarrow$) part, let X and Y be isomorphic Hilbert spaces. Let $U : X \longrightarrow Y$ be a unitary. Note that if $\{e_i\}_{i\in I}$ is an orthonormal basis of X, then $\{Ue_i\}_{i\in I}$ is also an orthonormal basis of Y. Thus the necessary part follows immediately from Proposition 14.9. □

Corollary 14.12. *Every separable Hilbert space is isomorphic to ℓ^2 or $\mathbb{C}^n$ for some n.*

Proof. Let X be a separable Hilbert space.
If $\dim X < \infty$, then it is clear that X is isomorphic to $\mathbb{C}^n$ for $n = \dim X$.
Now suppose that $\dim X = \infty$ and its orthonormal dimension is larger than $|\mathbb{N}|$, i.e., X has an orthonormal basis $\{f_i\}_{i\in I}$ with $|I| > |\mathbb{N}|$. Note that since $\|f_i - f_j\| = \sqrt{2}$ for all $i, j \in I$ with $i \neq j$. This implies that $B(f_i, 1/4) \cap B(f_j, 1/4) = \emptyset$ for $i \neq j$.
On the other hand, if we let D be a countable dense subset of X, then $B(f_i, 1/4) \cap D \neq \emptyset$ for all $i \in I$. Thusfor each $i \in I$, we can pick up an element $x_i \in D \cap B(f_i, 1/4)$. Therefore, one can define an injection from I into D. It is absurd to the countability of D. ☐

Example 14.13. The followings are important classes of Hilbert spaces.

(i) $\mathbb{C}^n$ is a n-dimensional Hilbert space. In this case , the inner product is given by $(z, w) := \sum_{k=1}^{n} z_k \overline{w}_k$ for $z = (z_1, ..., z_n)$ and $(w_1, ..., w_n)$ in $\mathbb{C}^n$.
The natural basis $\{e_1, ..., e_n\}$ forms an orthonormal basis for $\mathbb{C}^n$.

(ii) ℓ^2 is a separable Hilbert space of infinite dimension whose inner product is given by $(x, y) := \sum_{n=1}^{\infty} x(n)\overline{y(n)}$ for $x, y \in \ell^2$.
If we put $e_n(n) = 1$ and $e_n(k) = 0$ for $k \neq n$, then $\{e_n\}$ is an orthonormal basis for ℓ^2.

(iii) Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For each $f \in C(\mathbb{T})$ (the space of all complex-valued continuous functions defined on $\mathbb{T}$), the integral of f is defined by

$$\int_{\mathbb{T}} f(z)dz := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{Re}f(e^{it})dt + \frac{i}{2\pi}\int_0^{2\pi} \mathfrak{Im}f(e^{it})dt.$$

An inner product on $C(\mathbb{T})$ is given by

$$(f, g) := \int_{\mathbb{T}} f(z)\overline{g(z)}dz$$

for each $f, g \in C(\mathbb{T})$. We write $\|\cdot\|_2$ for the norm induced by this inner product.
The Hilbert space $L^2(\mathbb{T})$ is defined by the completion of $C(\mathbb{T})$ under the norm $\|\cdot\|_2$.
Now for each $n \in \mathbb{Z}$, put $f_n(z) = z^n$. We claim that $\{f_n : n = 0, \pm1, \pm2, ...\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
In fact, by using the Euler Formula: $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$, we see that the family $\{f_n : n \in \mathbb{Z}\}$ is orthonormal.
It remains to show that the family $\{f_n\}$ is maximal. By Proposition 14.8, it needs to show that if $(g, f_n) = 0$ for all $n \in \mathbb{Z}$, then $g = 0$ in $L^2(\mathbb{T})$. for showing this, we have to make use the known fact that every element in $L^2(\mathbb{T})$ can be approximated by the polynomial functions of z and $\bar{z}$ on $\mathbb{T}$ in $\|\cdot\|_2$-norm due to the *the Stone-Weierstrass Theorem:*

For a compact metric space E, suppose that a complex subalgebra A of $C(E)$ satisfies the conditions: (i): the conjugate $\bar{f} \in A$ whenever $f \in A$, (ii): for every pair $z, z' \in E$, there is $f \in A$ such that $f(z) \neq f(z')$ and (iii): A contains the constant one function. Then A is dense in $C(E)$ with respect to the sup*-norm.*

Thus, the algebra of all polynomials functions of z and $\bar{z}$ on $\mathbb{T}$ is dense in $C(\mathbb{T})$. From this we can find a sequence of polynomials $(p_n(z, \bar{z}))$ such that $\|g - p_n\|_2 \to 0$ as $n \to 0$. Since $(g, f_n) = 0$ for all n, we see that $(g, p_n) = 0$ for all n. Therefore, we have

$$\|g\|_2^2 = \lim_n (g, p_n) = 0.$$

The proof is complete.

Remark 14.14. In view of Example 14.13(*iii*) above, $L^2(\mathbb{T})$ can be identified as the space

$$L^2[0,2\pi] := \{f : [0,2\pi] \to \mathbb{C} : \ f \text{ is a Lebesgue measure function so that } \int_0^{2\pi} |f(x)|^2 dx \text{ exists}\}.$$

In this case, the inner product in $L^2[0,2\pi]$ is given by $(f,g) := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt$. Now for each $x \in [0,2\pi]$ and $f \in L^2[0,2\pi]$, put $e_k(x) := e^{ikx}$ and

$$S_N(f,x) := \sum_{k=-N}^{N} (f,e_k)e_k(x) = \frac{1}{2\pi}\sum_{k=-N}^{N}(\int_0^{2\pi} f(t)e^{-ik}dt)e^{ikx}. \tag{14.1}$$

Then the series $\sum_{-\infty}^{\infty}(f,e_k)e_k(x)$ (need not be convergent) is the usual notation of the Fourier series of f at x.

In the previous result, we have seen that the series $\lim_{N\to\infty} \|f - \sum_{k=-N}^{N}(f,e_k)e_k\|_{L^2} \to 0$

It is naturally raised the question: for $f \in C[0,2\pi]$, do we have $f(x) = \lim_{N\to\infty} S_N(f,x)$ for all $x \in [0,2\pi]$, that is, f can be written as its Fourier series? The answer is negative. Indeed, we have the following stronger result.

Theorem 14.15. *For each element $x_0 \in [0,2\pi]$, there is a continuous function f on $[0,2\pi]$ such that its Fourier series $\sum_{k=-\infty}^{\infty}(f,e_k)e^{ikx_0}$ at x_0 is divergent.*

Proof. We give an outline argument in here. The details of the proof is referred to the Katznelson's classic book [9].

We keep the notation as above. We fix a point $x_0 \in [0,2\pi]$. For $f \in C[0,2\pi]$, Using the Eq 14.1 above, we have

$$\begin{aligned} \varphi_N(f) := S_N(f,x_0) &= \frac{1}{2\pi}\sum_{k=-N}^{N}(\int_0^{2\pi} f(t)e^{-ik}dt)e^{ikx_0} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)(\sum_{k=-N}^{N} e^{ik(x_0-t)})dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)D_N(x_0-t)dt, \end{aligned} \tag{14.2}$$

where $D_N(s) := \sum_{k=-N}^{N} e^{iks}$ for $s \in \mathbb{R}$. Then $\varphi_N \in C[0,2\pi]^*$. It is a fact that we have (see [9, Chapter 2. §2])

$$\|\varphi_N\| = \frac{1}{2\pi}\int_0^{2\pi} |D_N(x_0-t)|dt \to \infty \quad \text{ as } N \to \infty.$$

From this we have $\sup\{\|\varphi_N\| : N = 1,2....\} = \infty$. This implies that the following set is dense in $C[0,2\pi]$.

$$D := \{f \in C[0,2\pi] : \sup_N |\varphi_N(f)| = \infty\}.$$

To see this, suppose not. Then there is an element $f_0 \in C[0,2\pi]$ and $r > 0$ such that $\sup_N |\varphi_N(f)| < \infty$ for all $f \in B(f_0,r)$. By considering $B(0,1) = \frac{1}{r}(B(f_0,r) - f_0)$, then we have $\sup_N |\varphi_N(g)| < \infty$ for all $g \in B(0,1)$ and so, $\sup_N |\varphi_N(g)| < \infty$ for all $g \in C[0,2\pi]$. The Uniform Boundedness Theorem implies that $\sup_N \|\varphi_N\| < \infty$ that leads to a contradiction.

On the other hand, it is clear that if the series $\sum_{k=-\infty}^{\infty}(f,e_k)e^{ikx_0}$ is convergent, then $\sup_N |S_N(f,x_0)| < \infty$. Therefore, the Fourier series of each element in D is divergent at x_0 as desired. We finish the proof. ☐

The above proof is an elegant application of functional analysis.

Students should be made to think, to doubt, to communicate, to question, to learn from their mistakes, and most importantly have fun in their learning.
∼ Richard Feynman ∼ one of the greatest physicists in 20th century.

15. Geometry of Hilbert space II

In this section, let X always denote a complex Hilbert space.

Proposition 15.1. *If D is a closed convex subset of X, then there is a unique element $z \in D$ such that*

$$\|z\| = \inf\{\|x\| : x \in D\}.$$

Consequently, for any element $u \in X$, there is a unique element $w \in D$ such that

$$\|u - w\| = d(u, D) := \inf\{\|u - x\| : x \in D\}.$$

Proof. We first claim the existence of such z.
Let $d := \inf\{\|x\| : x \in D\}$. Then there is a sequence (x_n) in D such that $\|x_n\| \to d$. Note that (x_n) is a Cauchy sequence. In fact, the Parallelogram Law implies that

$$\|\frac{x_m - x_n}{2}\|^2 = \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - \|\frac{x_m + x_n}{2}\|^2 \leq \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - d^2 \longrightarrow 0$$

as $m, n \to \infty$, where the last inequality holds because D is convex and hence $\frac{1}{2}(x_m + x_n) \in D$. Let $z := \lim_n x_n$. Then $\|z\| = d$ and $z \in D$ because D is closed.
For the uniqueness, let $z, z' \in D$ such that $\|z\| = \|z'\| = d$. Thanks to the Parallelogram Law again, we have

$$\|\frac{z - z'}{2}\|^2 = \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - \|\frac{z + z'}{2}\|^2 \leq \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - d^2 = 0.$$

Therefore $z = z'$.
The last assertion follows by considering the closed convex set $u - D := \{u - x : x \in D\}$ immediately. ☐

Remark 15.2. *Using the notation given as in Proposition 15.1, we have a well defined function $r : X \to X$ given by $x \in X \mapsto r(x) \in D$ such that $\|x - r(x)\| = dist(x, D)$. Clearly, we have $r(x) = x$ whenever $x \in D$. Moreover, we have the following assertion which are shown in* [5].

Proposition 15.3. *Using the notation as in Remark15.2, the map $r : X \to X$ is a contraction, hence, the map r is a* Lipschitz retraction *of D in X.*

Proof. We first claim that we have $Re(x - r(z), r(z) - z) \geq 0$ for all $x \in D$ and $z \in X$. In fact, let $z \in X$ and $x \in D$. Then by the definition, for all $t \in [0, 1]$ we have

$$\begin{aligned}
\|r(z) - z\|^2 &\leq \|z - tx - (1 - t)r(z)\|^2 \\
&= \|z - r(z) - t(x - r(z))\|^2 \\
&= \|z - r(z)\|^2 + t^2\|x - r(z)\|^2 + 2tRe(x - r(z), r(z) - z).
\end{aligned}$$

This gives $t^2\|x-r(z)\|^2+2tRe(x-r(z),r(z)-z)\geq 0$ for all $0\leq t\leq 1$. This implies that $Re(x-r(z),r(z)-z)\geq 0$ for all $x\in D$ and $z\in X$. From this, for $a,b\in X$ we have $Re(r(b)-r(a),r(a)-a)\geq 0$ and $Re(r(a)-r(b),r(b)-b)\geq 0$, so we have $Re(r(b)-r(a),r(a)-a)+Re(r(b)-r(a),b-r(b))\geq 0$. Thus, we have

$$\begin{aligned}\|r(b)-r(a)\|^2 &= Re(r(b)-r(a),r(b)-r(a))\\ &\leq Re(r(b)-r(a),b-a)\\ &\leq |(r(b)-r(a),b-a)|\\ &\leq \|r(b)-r(a)\|\|b-a\|.\end{aligned}$$

The proof is complete. □

Proposition 15.4. *Suppose that M is a closed subspace. Let $u\in X$ and $w\in M$. Then the followings are equivalent:*

(i): $\|u-w\|=d(u,M)$;

(ii): $u-w\perp M$, i.e., $(u-w,x)=0$ for all $x\in M$.

Consequently, for each element $u\in X$, there is a unique element $w\in M$ such that $u-w\perp M$.

Proof. Let $d:=d(u,M)$.
For proving $(i)\Rightarrow(ii)$, fix an element $x\in M$. Then for any $t>0$, note that since $w+tx\in M$, we have

$$d^2\leq\|u-w-tx\|^2=\|u-w\|^2+\|tx\|^2-2Re(u-w,tx)=d^2+\|tx\|^2-2Re(u-w,tx).$$

This implies that

$$2Re(u-w,x)\leq t\|x\|^2 \tag{15.1}$$

for all $t>0$ and for all $x\in M$. Thus by considering $-x$ in Eq.15.1, we obtain

$$2|Re(u-w,x)|\leq t\|x\|^2.$$

for all $t>0$. This implies that $Re(u-w,x)=0$ for all $x\in M$. Similarly, putting $\pm ix$ into Eq.15.1, we have $Im(u-w,x)=0$. Thus(ii) follows.
For $(ii)\Rightarrow(i)$, we need to show that $\|u-w\|^2\leq\|u-x\|^2$ for all $x\in M$. Note that since $u-w\perp M$ and $w\in M$, we have $u-w\perp w-x$ for all $x\in M$. This gives

$$\|u-x\|^2=\|(u-w)+(w-x)\|^2=\|u-w\|^2+\|w-x\|^2\geq\|u-w\|^2.$$

Part (i) follows.
The last statement is obtained immediately by Proposition 15.1. □

Theorem 15.5. *Let M be a closed subspace. Put*

$$M^\perp:=\{x\in X: x\perp M\}.$$

Then $M^\perp$ is a closed subspace and we have $X=M\oplus M^\perp$. Consequently, for $x\in X$ if $x=u\oplus v$ for $u\in M$ and $v\in M^\perp$, then $dist(x,M)=\|x-u\|$.
In this case, $M^\perp$ is called the orthogonal complement of M.

Proof. Clearly, $M^\perp$ is a closed subspace and $M\cap M^\perp=(0)$. We need to show $X=M+M^\perp$.
Let $u\in X$. Then by Proposition 15.4, we can find an element $w\in M$ such that $u-w\perp M$. Thus $u-w\in M^\perp$ and $u=w+(u-w)$.
The last assertion follows immediately from Proposition 15.4. The proof is complete. □

Corollary 15.6. *Let M be a closed subspace of X. Then $M\subsetneq X$ if and only if there is a non-zero element $z\in X$ such that $z\perp M$.*

Proof. It is clear from Theorem 15.5. □

Corollary 15.7. *If M is a closed subspace of X, then $M^{\perp\perp} = M$.*

Proof. Clearly, we have $M \subseteq M^{\perp\perp}$ by the definition of $M^{\perp\perp}$. Then M can be viewed as a closed subspace of the Hilbert space $M^{\perp\perp}$. Thus, if $M \subsetneq M^{\perp\perp}$, then there exists a non-zero element $z \in M^{\perp\perp}$ so that $z \perp M$ by Corollary 15.6 and hence, $z \in M^{\perp}$. This implies that $z \perp z$ and hence, $z = 0$ which leads to a contradiction. □

Theorem 15.8. Riesz Representation Theorem : *For each $f \in X^*$, then there is a unique element $v_f \in X$ such that*

$$f(x) = (x, v_f)$$

for all $x \in X$ and we have $\|f\| = \|v_f\|$.
Furthermore, if $(e_i)_{i\in I}$ is an orthonormal basis of X, then $v_f = \sum_i \overline{f(e_i)} e_i$.

Proof. We first prove the uniqueness of v_f. If $z \in X$ also satisfies the condition: $f(x) = (x, z)$ for all $x \in X$. This implies that $(x, z - v_f) = 0$ for all $x \in X$. Thus$z - v_f = 0$.
Now for proving the existence of v_f, it suffices to show the case $\|f\| = 1$. Then $\ker f$ is a closed proper subspace. Then by the orthogonal decomposition again, we have

$$X = \ker f \oplus (\ker f)^{\perp}.$$

Since $f \neq 0$, we have $(\ker f)^{\perp}$ is linear isomorphic to $\mathbb{C}$. Note that the restriction of f on $(\ker f)^{\perp}$ is of norm one. Hence there is an element $v_f \in (\ker f)^{\perp}$ with $\|v_f\| = 1$ such that $f(v_f) = \|f|_{(\ker f)^{\perp}}\| = 1$ and $(\ker f)^{\perp} = \mathbb{C} v_f$. Thusfor each element $x \in X$, we have $x = z + \alpha v_f$ for some $z \in \ker f$ and $\alpha \in \mathbb{C}$. Then $f(x) = \alpha f(v_f) = \alpha = (x, v_f)$ for all $x \in X$.
Concerning about the last assertion, if we put $v_f = \sum_{i\in I} \alpha_i e_i$, then $f(e_j) = (e_j, v_f) = \overline{\alpha_j}$ for all $j \in I$. □

Example 15.9. *Consider the Hilbert space $H := L^2(\mathbb{T})$ (see Example 14.13). Define $\varphi \in H^*$ by $\varphi(f) := \int_{\mathbb{T}} f(z)dz$. Using Proposition 15.4, for each element $g \in H$, there is an element $h \in \ker \varphi$ such that $\|g - h\| = dist(g, \ker \varphi)$. Then $h = g - (\int gdz)\mathbf{1}$ where $\mathbf{1}$ denotes the constant-one function on $\mathbb{T}$. In fact, consider the orthogonal decomposition $H = \ker \varphi \oplus (\ker \varphi)^{\perp}$. Note that $\varphi(g) = (g, \mathbf{1})$ for all $g \in H$. Thus, for each $g \in H$, we have $g = h \oplus \alpha \mathbf{1}$. From this, we see that $\alpha = (g, \mathbf{1})$. Thus, $h = g - (\int gdz)\mathbf{1}$.*

Corollary 15.10. *Using the notations as in Theorem 15.8, define the map*

$$\Phi : f \in X^* \mapsto v_f \in X, \ \text{i.e.,}\ f(y) = (x, \Phi(f)) \tag{15.2}$$

for all $y \in X$ and $f \in X^*$.
Moreover, if we define $(f, g)_{X^*} := (v_g, v_f)_X$ for $f, g \in X^*$, then $(X^*, (\cdot,\cdot)_{X^*})$ becomes a Hilbert space, and Φ is an anti-unitary operator from X^* onto X, i.e., Φ satisfies the conditions:

$$\Phi(\alpha f + \beta g) = \overline{\alpha}\Phi(f) + \overline{\beta}\Phi(g) \quad \text{and} \quad (\Phi f, \Phi g)_X = (g, f)_{X^*}$$

for all $f, g \in X^*$ and $\alpha, \beta \in \mathbb{C}$.
Furthermore, if we define $J : x \in X \mapsto f_x \in X^*$, where $f_x(y) := (y, x)$, then J is the inverse of Φ, and hence, J is an isometric conjugate linear isomorphism.

Proof. The result follows immediately from the observation that $v_{f+g} = v_f + v_g$ and $v_{\alpha f} = \overline{\alpha} v_f$ for all $f \in X^*$ and $\alpha \in \mathbb{C}$.
The last assertion is clearly obtained by the Eq.15.2 above. □

Corollary 15.11. *Every Hilbert space is reflexive.*

Proof. Using the notations as in the Riesz Representation Theorem 15.8, let X be a Hilbert space. and $Q : X \to X^{**}$ the canonical isometry. Let $\psi \in X^{**}$. To apply the Riesz Theorem on the dual space X^*, there exists an element $x_0^* \in X^*$ such that

$$\psi(f) = (f, x_0^*)_{X^*}$$

for all $f \in X^*$. By using Corollary 15.10, there is an element $x_0 \in X$ such that $x_0 = v_{x_0^*}$ and thus, we have

$$\psi(f) = (f, x_0^*)_{X^*} = (x_0, v_f)_X = f(x_0)$$

for all $f \in X^*$. Therefore, $\psi = Q(x_0)$ and so, X is reflexive.
The proof is complete. □

Theorem 15.12. *Every bounded sequence in a Hilbert space has a weakly convergent subsequence.*

Proof. Let (x_n) be a bounded sequence in a Hilbert space X and M be the closed subspace of X spanned by $\{x_m : m = 1, 2...\}$. Then M is a separable Hilbert space.
Method I : Define a map by $j_M : x \in M \mapsto j_M(x) := (\cdot, x) \in M^*$. Then $(j_M(x_n))$ is a bounded sequence in M^*. By Banach's result, Proposition 6.9, $(j_M(x_n))$ has a w^*-convergent subsequence $(j_M(x_{n_k}))$. Put $j_M(x_{n_k}) \xrightarrow{w^*} f \in M^*$, i.e., $j_M(x_{n_k})(z) \to f(z)$ for all $z \in M$. The Riesz Representation will assure that there is a unique element $m \in M$ such that $j_M(m) = f$. Thuswe have $(z, x_{n_k}) \to (z, m)$ for all $z \in M$. In particular, if we consider the orthogonal decomposition $X = M \oplus M^\perp$, then $(x, x_{n_k}) \to (x, m)$ for all $x \in X$ and thus $(x_{n_k}, x) \to (m, x)$ for all $x \in X$. Then $x_{n_k} \to m$ weakly in X by using the Riesz Representation Theorem again.
Method II : We first note that since M is a separable Hilbert space, the second dual M^{**} is also separable by the reflexivity of M. Thus, the dual space M^* is separable (see Proposition4.11). Let $Q : M \longrightarrow M^{**}$ be the natural canonical mapping. To apply the Banach's result Proposition 6.9 for X^*, then $Q(x_n)$ has a w^*-convergent subsequence, says $Q(x_{n_k})$. This gives an element $m \in M$ such that $Q(m) = w^*\text{-}\lim_k Q(x_{n_k})$ because M is reflexive. Thus, we have $f(x_{n_k}) = Q(x_{n_k})(f) \to Q(m)(f) = f(m)$ for all $f \in M^*$. Using the same argument as in **Method I** again, x_{n_k} weakly converges to m. □

Remark 15.13. Recall the well known James's Theorem that a Banach space X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence. (see Appendix 7 for the proof of a separable space case). Hence, Theorem 15.12 can be obtained by the James's Theorem directly. However, the Riesz Representation Theorem gives a simple proof for the Hilbert spaces case.

16. Operators on a Hilbert space

Throughout this section, all spaces are complex Hilbert spaces. Let $B(X, Y)$ denote the space of all bounded linear operators from X into Y. If $X = Y$, we write $B(X)$ for $B(X, X)$.
Let $T \in B(X, Y)$. We make use the following simple observation later.

$$(Tx, y) = 0 \text{ for all } x \in X; y \in Y \quad \text{if and only if} \quad T = 0. \tag{16.1}$$

Therefore, the elements in $B(X, Y)$ are uniquely determined by the Eq.16.1, i.e., $T = S$ in $B(X, Y)$ if and only if $(Tx, y) = (Sx, y)$ for all $x \in X$ and $y \in Y$.

Remark 16.1. *For Hilbert spaces H_1 and H_2, we consider their direct sum $H := H_1 \oplus H_2$. If we define the inner product on H by*

$$(x_1 \oplus x_2, y_1 \oplus y_2) := (x_1, y_1)_{H_1} + (x_2, y_2)_{H_2}$$

for $x_1 \oplus x_2$ and $y_1 \oplus y_2$ in H, then H becomes a Hilbert space. Now for each $T \in B(H_1, H_2)$, we can define an element $\tilde{T} \in B(H)$ by $\tilde{T}(x_1 \oplus x_2) := 0 \oplus Tx_1$. Therefore, the space $B(H_1, H_2)$ can be viewed as a closed subspace of $B(H)$. Thus, we can consider the case of $H_1 = H_2$ for studying the space $B(H_1, H_2)$.

Proposition 16.2. *Let $T : X \to X$ be a linear operator. Then we have*

- *(i): $T = 0$ if and only if $(Tx, x) = 0$ for all $x \in X$. Consequently, for $T, S \in B(X)$, $T = S$ if and only if $(Tx, x) = (Sx, x)$ for all $x \in X$.*
- *(ii): T is bounded if and only if $\sup\{|(Tx, y)| : x, y \in X$ with $\|x\| = \|y\| = 1\}$ is finite. In this case, we have $\|T\| = \sup\{|(Tx, y)| : x, y \in X$ with $\|x\| = \|y\| = 1\}$.*

Proof. Clearly, the necessary part holds in Part (i). We want to show the sufficient part in Part (i). We assume that $(Tx, x) = 0$ for all $x \in X$. Then we have

$$0 = (T(x + iy), x + iy) = (Tx, x) + i(Ty, x) - i(Tx, y) + (Tiy, iy) = i(Ty, x) - i(Tx, y).$$

Thus, we have $(Ty, x) - (Tx, y) = 0$ for all $x, y \in X$. In particular, if we replace y by iy in the equation, then we get $i(Ty, x) - \bar{i}(Tx, y) = 0$ and hence we have $(Ty, x) + (Tx, y) = 0$. Therefore we have $(Tx, y) = 0$.
For showing part (ii), let $\alpha = \sup\{|(Tx, y)| : x, y \in X$ with $\|x\| = \|y\| = 1\}$. It suffices to show $\|T\| = \alpha$. Clearly, we have $\|T\| \geq \alpha$. We need to show $\|T\| \leq \alpha$.
In fact, let $x \in X$ with $\|x\| = 1$. If $Tx \neq 0$, then we take $y = Tx/\|Tx\|$. Thus, we have $\|Tx\| = |\big(Tx, y\big)| \leq \alpha$, and so $\|T\| \leq \alpha$. The proof is complete. □

Proposition 16.3. *Let $T \in B(X)$. Then there is a unique element T^* in $B(X)$ such that*

$$(Tx, y) = (x, T^*y) \tag{16.2}$$

In this case, T^* is called the adjoint operator of T.

Proof. First, we show the uniqueness. Suppose that there are S_1, S_2 in $B(X)$ which satisfy the Eq.16.2. Then $(x, S_1y) = (x, S_2y)$ for all $x, y \in X$. Eq.16.1 implies that $S_1 = S_2$.
Finally, we prove the existence. Note that if we fix an element $y \in X$, define the map $f_y(x) := (Tx, y)$ for all $x \in X$. Then $f_y \in X^*$. By applying the Riesz Representation Theorem, there is a unique element $y^* \in X$ such that $(Tx, y) = (x, y^*)$ for all $x \in X$ and $\|f_y\| = \|y^*\|$. In addition, we have

$$|f_y(x)| = |(Tx, y)| \leq \|T\|\|x\|\|y\|$$

for all $x, y \in X$ and thus $\|f_y\| \leq \|T\|\|y\|$. If we put $T^*(y) := y^*$, then T^* satisfies the Eq.16.2. Moreover, we have $\|T^*y\| = \|y^*\| = \|f_y\| \leq \|T\|\|y\|$ for all $y \in X$. Thus, we have $T^* \in B(X)$ and $\|T^*\| \leq \|T\|$. Hence the operator T^* is as desired. □

Remark 16.4. Let $S, T : X \to X$ be linear operators (without assuming to be bounded). If they satisfy the Eq.16.2 above, i.e.,

$$(Tx, y) = (x, Sy)$$

for all $x, y \in X$. Using the Closed Graph Theorem, we can show that S and T both are automatically bounded.
In fact, let (x_n) be a sequence in X such that $\lim x_n = x$ and $\lim Sx_n = y$ for some $x, y \in X$. Now for any $z \in X$, we have

$$(z, Sx) = (Tz, x) = \lim(Tz, x_n) = \lim(z, Sx_n) = (z, y).$$

Thus $Sx = y$ and hence S is bounded by the Closed Graph Theorem.
Similarly, we can also see that T is bounded.

Remark 16.5. Let $T \in B(X)$. Let $T^t : X^* \to X^*$ be the transpose of T which is defined by $T^t(f) := f \circ T \in X^*$ for $f \in X^*$ (see Proposition 4.13). Then we have the following commutative diagram (**Check!**)

$$\begin{array}{ccc} X & \xrightarrow{T^*} & X \\ J_X \big\downarrow & & \big\downarrow J_X \\ X^* & \xrightarrow{T^t} & X^* \end{array}$$

where $J_X : X \to X^*$ is the anti-unitary given by the Riesz Representation Theorem (see Corollary 15.10).

Proposition 16.6. *Let $T, S \in B(X)$. Then we have*

(i): $T^* \in B(X)$ and $\|T^*\| = \|T\|$.

(ii): The map $T \in B(X) \mapsto T^* \in B(X)$ is an isometric conjugate anti-isomorphism, i.e.,

$$(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^* \quad \textit{for all} \quad \alpha, \beta \in \mathbb{C}; \quad \textit{and} \quad (TS)^* = S^* T^*.$$

(iii): $\|T^* T\| = \|T\|^2$.

Proof. For Part (i), in the proof of Proposition 16.3, we have shown that $\|T^*\| \le \|T\|$. In addition, the reverse inequality follows clearly from $T^{**} = T$.
The Part (ii) follows from the adjoint operators which are uniquely determined by the Eq.16.2 above.
For Part (iii), we always have $\|T^* T\| \le \|T^*\| \|T\| = \|T\|^2$. For the reverse inequality, let $x \in B_X$. Then

$$\|Tx\|^2 = (Tx, Tx) = (T^* T x, x) \le \|T^* T x\| \|x\| \le \|T^* T\|.$$

Therefore, we have $\|T\|^2 \le \|T^* T\|$. ☐

Example 16.7. *If $X = \mathbb{C}^n$ and $D = (a_{ij})_{n\times n}$ an $n \times n$ matrix, then $D^* = (\overline{a_{ji}})_{n\times n}$. In fact, note that*

$$a_{ji} = (De_i, e_j) = (e_i, D^* e_j) = \overline{(D^* e_j, e_i)}.$$

Thusif we put $D^* = (d_{ij})_{n\times n}$, then $d_{ij} = (D^* e_j, e_i) = \overline{a_{ji}}$.

Example 16.8. *Let $\ell^2(\mathbb{N}) := \{x : \mathbb{N} \to \mathbb{C} : \sum_{i=0}^\infty |x(i)|^2 < \infty\}$, and put $(x, y) := \sum\limits_{i=0}^{\infty} x(i)\overline{y(i)}$.*

Define the operator $D \in B(\ell^2(\mathbb{N}))$ (called the unilateral shift) by

$$Dx(i) = x(i-1)$$

for $i \in \mathbb{N}$, where we set $x(-1) := 0$, i.e., $D(x(0), x(1), ...) = (0, x(0), x(1), ...)$.
Then D is an isometry and the adjoint operator D^* is given by

$$D^* x(i) := x(i+1)$$

for $i = 0, 1, .., $ i.e., $D^*(x(0), x(1), ...) = (x(1), x(2), ...)$.
Indeed we can directly check that

$$(Dx, y) = \sum_{i=0}^{\infty} x(i-1)\overline{y(i)} = \sum_{j=0}^{\infty} x(j)\overline{y(j+1)} = (x, D^* y).$$

Note that D^* is NOT an isometry.

Example 16.9. *Let* $\ell^\infty(\mathbb{N}) = \{x : \mathbb{N} \to \mathbb{C} : \sup_{i\geq 0} |x(i)| < \infty\}$ *and* $\|x\|_\infty := \sup_{i\geq 0} |x(i)|$. *For each* $x \in \ell^\infty$, *define* $M_x \in B(\ell^2(\mathbb{N}))$ *by*

$$M_x(\xi) := x \cdot \xi$$

for $\xi \in \ell^2(\mathbb{N})$, *where* $(x \cdot \xi)(i) := x(i)\xi(i)$; $i \in \mathbb{N}$.
Then $\|M_x\| = \|x\|_\infty$ *and* $M_x^* = M_{\overline{x}}$, *where* $\overline{x}(i) := \overline{x(i)}$.

Definition 16.10. *Let* $T \in B(X)$ *and let* I *be the identity operator on* X. T *is said to be*

(i) : selfadjoint if $T^* = T$;
(ii) : normal if $T^*T = TT^*$;
(iii) : unitary if $T^*T = TT^* = I$.

Proposition 16.11. *We have*

(i) : Let $T : X \longrightarrow X$ *be a linear operator.* T *is a bounded linear selfadjoint operator if and only if we have*

$$(Tx, y) = (x, Ty) \quad \text{for all } x, y \in X. \tag{16.3}$$

(ii) : T *is normal if and only if* $\|Tx\| = \|T^*x\|$ *for all* $x \in X$.

Proof. The necessary part of Part (i) is clear.
Now suppose that the Eq.16.3 holds, it needs to show that T is bounded. Indeed, it follows immediately from Remark16.4.
For Part (ii), note that by Proposition 16.2, T is normal if and only if $(T^*Tx, x) = (TT^*x, x)$. Thus, Part (ii) follows from

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (TT^*x, x) = (T^*x, T^*x) = \|T^*x\|^2$$

for all $x \in X$. □

Remark 16.12. *In Proposition 16.11(i), if the domain of* T *is replaced by dense domain, then the conclusion does not hold. For example, let* $D := \{x \in \ell^2 : \sum_{n=1}^\infty |nx(n)|^2 < \infty\}$ *and let* $T(x)(n) := nx(n)$ *for* $x \in D$. *Then* D *is a dense domain because the canonical basis* $(e_n) \subseteq D$. *It is noted that* T *is unbounded on* D, *but* $(Tx, y) = (x, Ty)$ *for all* $x, y \in D$.

Proposition 16.13. *Let* $T \in B(H)$. *We have the following assertions.*

(i) : T *is selfadjoint if and only if* $(Tx, x) \in \mathbb{R}$ *for all* $x \in H$.
(ii) : If T *is selfadjoint, then* $\|T\| = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$.

Proof. Part (i) follows immediately from Proposition16.2.
For Part (ii), if we let $a = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$, then we have $a \leq \|T\|$. We want to show the reverse inequality. T is selfadjoint, and so we can directly check that

$$(T(x + y), x + y) - (T(x - y), x - y) = 4Re(Tx, y)$$

for all $x, y \in H$. Thus if $x, y \in H$ with $\|x\| = \|y\| = 1$ and $(Tx, y) \in \mathbb{R}$, then by using the Parallelogram Law, we have

$$|(Tx, y)| \leq \frac{a}{4}(\|x + y\|^2 + \|x - y\|^2) = \frac{a}{2}(\|x\|^2 + \|y\|^2) = a. \tag{16.4}$$

Now for $x, y \in H$ with $\|x\| = \|y\| = 1$, by considering the polar form of $(Tx, y) = re^{i\theta}$, the Eq.16.4 gives

$$|(Tx, y)| = |(Tx, e^{i\theta}y)| \leq a.$$

$\|T\| = \sup\limits_{\|x\|=\|y\|=1} |(Tx, y)|$, and so we have $\|T\| \leq a$. The proof is complete. □

Proposition 16.14. *Let $T \in B(X)$. Then we have*

$$\ker T = (imT^*)^\perp \quad \text{and} \quad (\ker T)^\perp = \overline{imT^*}$$

where imT denotes the image of T.

Proof. The first equality follows clearly from $x \in \ker T$ if and only if $0 = (Tx, z) = (x, T^*z)$ for all $z \in X$.
On the other hand, it is clear that we have $M^\perp = \overline{M}^\perp$ for any subspace M of X. This, together with the first equality and Corollary15.7, gives immediately the second equality. ☐

Proposition 16.15. *Let X be a Hilbert space. Let M and N be the closed subspaces of X such that*

$$X = M \oplus N \qquad \ldots\ldots\ldots\ldots(*)$$

Let $Q : X \to X$ be the projection along the decomposition $(*)$ with im $Q = M$ (note that Q is bounded by Proposition 12.1). Then $N = M^\perp$ (and hence $(*)$ is the orthogonal decomposition of X with respect to M) if and only if Q satisfies the conditions: $Q^2 = Q$ and $Q^* = Q$. In this case, Q is called the orthogonal projection (or projection for simply) with respect to M.

Proof. Now if $N = M^\perp$, then for $y, y' \in M$ and $z, z' \in N$, we have

$$(Q(y + z), y' + z') = (y, y') = (y + z, Q(y' + z')).$$

Thus$Q^* = Q$.
The converse of the last statement follows immediately from Proposition 16.14 because $\ker Q = N$ and $imQ = M$.
The proof is complete. ☐

Proposition 16.16. *When X is a Hilbert space, we put $\mathcal{M}$ the set of all closed subspaces of X and $\mathcal{P}$ the set of all orthogonal projections on X. Now for each $M \in \mathcal{M}$, let P_M be the corresponding projection with respect to the orthogonal decomposition $X = M \oplus M^\perp$. Then there is an one-one correspondence between $\mathcal{M}$ and $\mathcal{P}$ which is defined by*

$$M \in \mathcal{M} \mapsto P_M \in \mathcal{P}.$$

Furthermore, if $M, N \in \mathcal{M}$, then we have

- *(i) : $M \subseteq N$ if and only if $P_M P_N = P_N P_M = P_M$.*
- *(ii) : $M \perp N$ if and only if $P_M P_N = P_N P_M = 0$.*

Proof. Using Proposition 16.15, we note that $P_M \in \mathcal{P}$.
Indeed the inverse of the correspondence is given by the following. If we let $Q \in \mathcal{P}$ and $M = Q(X)$, then M is closed. In addition, clearly we have $X = Q(X) \oplus (I - Q)X$ with $M^\perp = (I - Q)X$. Hence M is the corresponding closed subspace of X, i.e., $M \in \mathcal{M}$ and $P_M = Q$.
For the final assertion, Part (i) and (ii) follow immediately from the orthogonal decompositions $X = M \oplus M^\perp = N \oplus N^\perp$ and together with the fact that $M \subseteq N$ if and only if $N^\perp \subseteq M^\perp$. ☐

17. Spectral Theory I

Definition 17.1. *Let E be a normed space and let $T \in B(E)$. The spectrum of T, denoted by $\sigma(T)$, is defined by*

$$\sigma(T) := \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not invertible in } B(E)\}.$$

Remark 17.2. *More precisely, for a normed space E, an operator $T \in B(E)$ is said to be invertible in $B(E)$ if T is an linear isomorphism and the inverse T^{-1} is also bounded. However, if E is complete, the Open Mapping Theorem assures that the inverse T^{-1} is bounded automatically. Thus if E is a Banach space and $T \in B(E)$, then $\lambda \notin \sigma(T)$ if and only if $T - \lambda := T - \lambda I$ is an linear isomorphism. Thus, λ lies in the spectrum $\sigma(T)$ if and only if $T - \lambda$ is either not one-one or not surjective.*
In particular, if there is a non-zero element $v \in X$ such that $Tv = \lambda v$, then $\lambda \in \sigma(T)$ and λ is called an eigenvalue of T with eigenvector v.
In addition, we write $\sigma_p(T)$ for the set of all eigenvalue of T and call $\sigma_p(T)$ the point spectrum.

Example 17.3. *Let $E = \mathbb{C}^n$ and $T = (a_{ij})_{n\times n} \in M_n(\mathbb{C})$. Then $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue of T and thus $\sigma(T) = \sigma_p(T)$.*

Example 17.4. Let $E = (c_{00}(\mathbb{N}), \|\cdot\|_\infty)$ (note that $c_{00}(\mathbb{N})$ is not a Banach space). Define the map $T : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$ by

$$Tx(k) := \frac{x(k)}{k+1}$$

for $x \in c_{00}(\mathbb{N})$ and $i \in \mathbb{N}$.
Then T is bounded, in fact, $\|Tx\|_\infty \le \|x\|_\infty$ for all $x \in c_{00}(\mathbb{N})$.
On the other hand, we note that if $\lambda \in \mathbb{C}$ and $x \in c_{00}(\mathbb{N})$, then

$$(T - \lambda)x(k) = (\frac{1}{k+1} - \lambda)x(k).$$

From this we see that $\sigma_p(T) = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. In addition, if $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$, then $T - \lambda$ is an linear isomorphism and its inverse is given by

$$(T - \lambda)^{-1}x(k) = (\frac{1}{k+1} - \lambda)^{-1}x(k).$$

Thus, $(T - \lambda)^{-1}$ is unbounded if $\lambda = 0$,so $0 \in \sigma(T)$.
Besides, if $\lambda \notin \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$, then $(T - \lambda)^{-1}$ is bounded. In fact, if $\lambda = a + ib \neq 0$, for $a, b \in \mathbb{R}$, then $\eta := \min_k |\frac{1}{1+k} - a|^2 + |b|^2 > 0$ because $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. This gives

$$\|(T - \lambda)^{-1}\| = \sup_{k\in\mathbb{N}} |(\frac{1}{k+1} - \lambda)^{-1}| < \eta^{-1} < \infty.$$

We can now conclude that $\sigma(T) = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$.

Proposition 17.5. *Let E be a Banach space and $T \in B(E)$. Then*

- *(i) : $I - T$ is invertible in $B(E)$ whenever $\|T\| < 1$.*
- *(ii) : If $|\lambda| > \|T\|$, then $\lambda \notin \sigma(T)$.*
- *(iii) : $\sigma(T)$ is a compact subset of $\mathbb{C}$.*
- *(iv) : If we let $GL(E)$ the set of all invertible elements in $B(E)$, then $GL(E)$ is an open subset of $B(E)$ with respect to the $\|\cdot\|$-topology. Moreover, the map $T \in GL(E) \mapsto T^{-1} \in GL(E)$ is continuous in the norm-topology.*

Proof. Note that since $B(E)$ is complete, Part (i) follows immediately from the following equality.

$$(I - T)(I + T + T^2 + \cdots\cdots + T^{N-1}) = I - T^N$$

for all $N \in \mathbb{N}$.
For Part (ii), if $|\lambda| > \|T\|$, then by Part (i) , we see that $I - \frac{1}{\lambda}T$ is invertible and so is $\lambda I - T$. This implies $\lambda \notin \sigma(T)$.
For Part (iii), since $\sigma(T)$ is bounded by Part (ii), we need to show that $\sigma(T)$ is closed.

Let $c \in \mathbb{C} \setminus \sigma(T)$. We need to find $r > 0$ such that $\mu \notin \sigma(T)$ as $|\mu - c| < r$. Note that since $T - c$ is invertible, then for $\mu \in \mathbb{C}$, we have $T - \mu = (T - c) - (\mu - c) = (T - c)(I - (\mu - c)(T - c)^{-1})$. Therefore, if $\|(\mu - c)(T - c)^{-1})\| < 1$, then $T - \mu$ is invertible by Part (i). Thus, if we take $0 < r < \dfrac{1}{\|(T-c)^{-1}\|}$, then r is as desired, i.e., $B(c, r) \subseteq \mathbb{C} \setminus \sigma(T)$. Hence $\sigma(T)$ is closed.
For the last assertion, let $T \in GL(E)$. Note that for any $S \in B(E)$, we have $S = S - T + T = T(1 - T^{-1}(T - S))$. Thus, if $1 - T^{-1}(T - S)$ is invertible, then so is S. Using Part (i), if $\|T - S\| < 1/\|T^{-1}\|$, then $1 - T^{-1}(T - S)$ is invertible. Therefore we have $B(T, \frac{1}{\|T^{-1}\|}) \subseteq GL(E)$.
Finally, we show the inverse map is continuous. It suffices to show that if (T_n) is a sequence in $GL(E)$ so that $T_n \to I$, then $T_n^{-1} \to 1$. Note that if $\|T_n - 1\| < 1/2$, then $T_n^{-1} = \sum_{k=0}^{\infty}(1 - T_n)^k$, hence, we may assume that (T_n^{-1}) is uniformly bounded by 2. Therefore,

$$\|T_n^{-1} - 1\| \le \|T_n^{-1}\|\|T_n - 1\| \le 2\|T_n - 1\|.$$

The proof is complete. □

Corollary 17.6. *If U is a unitary operator on a Hilbert space X, then $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*

Proof. Since $\|U\| = 1$, we have $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ by Proposition 17.5(ii).
Now if $|\lambda| < 1$, then $\|\lambda U^*\| < 1$. By using Proposition 17.5 again, we have $I - \lambda U^*$ is invertible. This implies that $U - \lambda = U(I - \lambda U^*)$ is invertible and thus $\lambda \notin \sigma(U)$. □

Example 17.7. Let $E = \ell^2(\mathbb{N})$ and let $D \in B(E)$ be the right unilateral shift operator as in Example16.8. Recall that $Dx(k) := x(k - 1)$ for $k \in \mathbb{N}$ and $x(-1) := 0$. Then $\sigma_p(D) = \emptyset$ and $\sigma(D) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.
We first claim that $\sigma_p(D) = \emptyset$.
Suppose that $\lambda \in \mathbb{C}$ and $x \in \ell^2(\mathbb{N})$ satisfy the equation $Dx = \lambda x$. Then by the definition of D, we have

$$x(k - 1) = \lambda x(k) \qquad \cdots\cdots\cdots (*)$$

for all $k \in \mathbb{N}$.
If $\lambda \ne 0$, then we have $x(k) = \lambda^{-1} x_{k-1}$ for all $k \in \mathbb{N}$. Since $x(-1) = 0$, this forces $x(k) = 0$ for all i, i.e., $x = 0$ in $\ell^2(\mathbb{N})$.
On the other hand if $\lambda = 0$, the Eq.$(*)$ gives $x(k - 1) = 0$ for all k and so $x = 0$ again.
Therefore $\sigma_p(D) = \emptyset$.
Finally, we are going to show $\sigma(D) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.
Note that since D is an isometry, $\|D\| = 1$. Proposition 17.5 tells us that

$$\sigma(D) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le 1\}.$$

Note that since $\sigma_p(D)$ is empty, it suffices to show that $D - \mu$ is not surjective for all $\mu \in \mathbb{C}$ with $|\mu| \le 1$.
Now suppose that there is $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ such that $D - \lambda$ is surjective.
We consider the case where $|\lambda| = 1$ first.
Let $e_1 = (1, 0, 0, ...) \in \ell^2(\mathbb{N})$. Then by the assumption, there is $x \in \ell^2(\mathbb{N})$ such that $(D - \lambda)x = e_1$ and thus $Dx = \lambda x + e_1$. This implies that

$$x(k - 1) = Dx(k) = \lambda x(k) + e_1(k)$$

for all $k \in \mathbb{N}$. From this we have $x(0) = -\lambda^{-1}$ and $x(k) = -\lambda^{-k} x(0)$ for all $k \ge 1$ because $e_1(0) = 1$ and $e_1(k) = 0$ for all $k \ge 1$. Moreover, since $|\lambda| = 1$, it turns out that $|x(0)| = |x(k)|$ for all $k \ge 1$. As $x \in \ell^2(\mathbb{N})$, this forces $x = 0$. However, it is absurd because $Dx = \lambda x + e_1$.
Now we consider the case where $|\lambda| < 1$.
By Proposition 16.14, we have

$$\overline{im(D - \lambda)}^{\perp} = \ker(D - \lambda)^* = \ker(D^* - \overline{\lambda}).$$

Thus if $D - \lambda$ is surjective, we have $\ker(D^* - \overline{\lambda}) = (0)$ and hence $\overline{\lambda} \notin \sigma_p(D^*)$.
Note that the adjoint D^* of D is given by the left shift operator, i.e.,

$$D^*x(k) = x(k+1) \qquad \cdots\cdots\cdots(**)$$

for all $k \in \mathbb{N}$.
Now when $D^*x = \mu x$ for some $\mu \in \mathbb{C}$ and $x \in \ell^2(\mathbb{N})$, by using Eq.$(**)$, which is equivalent to saying that

$$x(k+1) = \mu x(k)$$

for all $k \in \mathbb{N}$. Therefore, if $|\overline{\lambda}| = |\lambda| < 1$ and we set $x(0) = 1$ and $x(k+1) = \overline{\lambda}^k x(0)$ for all $k \geq 1$, then $x \in \ell^2(\mathbb{N})$ and $D^*x = \overline{\lambda}x$. Hence $\overline{\lambda} \in \sigma_p(D^*)$ which leads to a contradiction. The proof is complete.

18. Spectral Theory II

Throughout this section, let H be a complex Hilbert space.

Lemma 18.1. *Let $T \in B(H)$ be a normal operator (recall that $T^*T = TT^*$). Then T is invertible in $B(H)$ if and only if there is $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in H$.*

Proof. The necessary part is obvious.
Now we want to show the converse. We first show the case where T is selfadjoint. Clearly, T is injective from the assumption. By the Open Mapping Theorem, we need to show that T is surjective.
In fact since $\ker T = \overline{imT^*}^{\perp}$ and $T = T^*$, we see that the image of T is dense in H.
Now if $y \in H$, then there is a sequence (x_n) in H such that $Tx_n \to y$. Thus, (Tx_n) is a Cauchy sequence. From this and the assumption give us that (x_n) is also a Cauchy sequence. If x_n converges to $x \in H$, then $y = Tx$. Therefore the assertion is true when T is selfadjoint.
Now if T is normal, then we have $\|T^*x\| = \|Tx\| \geq c\|x\|$ for all $x \in H$ by Proposition 16.11(ii). Therefore, we have $\|T^*Tx\| \geq c\|Tx\| \geq c^2\|x\|$. Hence T^*T still satisfies the assumption. Note that T^*T is selfadjoint. Therefore, we can apply the previous case to know that T^*T is invertible. This implies that T is also invertible because $T^*T = TT^*$.
The proof is complete. □

Definition 18.2. *Let $T \in B(H)$. We say that T is positive, denoted by $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$. For a pair of selfadjoint operators S and T, we say that $S \leq T$ if $T - S \geq 0$.*

Remark 18.3. *Clearly, a positive operator is selfadjoint by Proposition 16.13.*
In particular, all projections are positive.

Proposition 18.4. *If T is an invertible operator in $B(H)$, then the inverse T^{-1} of T belongs to the closed $*$-subalgebra of $B(H)$ generated by T and I.*

Proof. Put $S := T^*T$. Then S is invertible in $B(H)$. Now we may assume that $\|S\| \leq 1$. Lemma 18.1 gives $c > 0$ such that $(x,x) \geq (S^2x, x) \geq c(x,x)$ for all $x \in H$. We choose a positive integer N such that $Nc \geq 1$. Then we have

$$(x,x) \geq \frac{1}{Nc}(x,x) \geq \frac{1}{Nc}(S^2x,x) \geq \frac{1}{N}(x,x)$$

for all $x \in H$. Thus, we have

$$0 \leq I - \frac{S^2}{Nc} \leq I - \frac{1}{N}I < I.$$

If we let $R := I - \frac{S^2}{Nc}$, then $(I-R)^{-1}$ exists in $B(H)$ and hence we have

$$(\frac{S^2}{Nc})^{-1} = (I-R)^{-1} = \sum_{n=0}^{\infty}(I - \frac{S^2}{Nc})^n.$$

Then the result follows from

$$T^{-1} = \frac{1}{Nc}\sum_{n=0}^{\infty}(I - \frac{(T^*T)^2}{Nc})^n T^*TT^*.$$

□

Proposition 18.5. *Let $T \in B(H)$. We have*

(i) : If $T \geq 0$, then $T+I$ is invertible.

(ii) : If T is self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$. In particular, if $T \geq 0$, we have $\sigma(T) \subseteq [0,\infty)$.

Proof. For Part (i), we assume that $T \geq 0$. This implies that

$$\|(I+T)x\|^2 = \|x\|^2 + \|Tx\|^2 + 2(Tx,x) \geq \|x\|^2$$

for all $x \in H$. Thus, the invertibility of $I+T$ follows from Lemma 18.1.
For Part (ii), we first claim that $T+i$ is invertible. Indeed, it follows immediately from $(T+i)^*(T+i) = T^2+I$ and Part (i).
Now if $\lambda = a+ib$ where $a,b \in \mathbb{R}$ with $b \neq 0$, then $T-\lambda = -b(\frac{-1}{b}(T-a)+i)$ is invertible because $\frac{-1}{b}(T-a)$ is selfadjoint. Thus, $\sigma(T) \subseteq \mathbb{R}$.
Finally we want to show $\sigma(T) \subseteq [0,\infty)$ when $T \geq 0$. Note that since $\sigma(T) \subseteq \mathbb{R}$, it suffices to show that $T-c$ is invertible if $c<0$. Indeed, if $c<0$, then we see that $T-c = -c(I+(\frac{-1}{c}T))$ is invertible by the previous assertion because $\frac{-1}{c}T \geq 0$.
The proof is complete. □

Remark 18.6. *In Proposition 18.5, we have shown that if T is selfadjoint, then $\sigma(T) \subseteq \mathbb{R}$. However, the converse does not hold. For example, consider $H = \mathbb{C}^2$ and*

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Example 18.7. *Notice that the multiplication defines an isometry $M : x \in \ell_\infty \mapsto M(x) \in B(\ell_2)$ by $M(x)(\xi)(n) := x(n)\xi(n); n = 1,2...$ for $\xi \in \ell_2$. Then $M(\bar{x}) = M(x)^*$ for $x \in \ell_\infty$, and so, $M(x)$ is self-adjoint if and only if x is a $\mathbb{R}$-sequence. Now let $x \in \ell_\infty$ be a $\mathbb{R}$-sequence. For simply for each element $x \in \ell_\infty$, we also write x for $M(x)$ as an element in $B(\ell_2)$.*
Now we claim that if $x \in \ell_\infty$ is self-adjoint, then $\lambda \in \sigma(x)$ if and only if $\inf_n |x(n)-\lambda| = 0$. Consequently, $\sigma_p(x) = \{x_n : n = 1,2...\}$ and $\sigma(x) = \overline{\{x(n) : n = 1,2....\}}$.
In fact, for showing ($\Leftarrow$), let $\lambda \in \mathbb{R}$ such that $\inf_n |x(n)-\lambda| = 0$. If $x-\lambda$ is invertible in $B(\ell_2)$, then by Lemma 18.1, there is $c>0$ such that $\|(x-\lambda)\xi\| \geq c$ for all $\xi \in \ell_2$ of norm one. In particular, for each $n = 1,2...$, we have $|x(n)-\lambda| = \|(x-\lambda)(e_n)\| \geq c > 0$. It leads to a contradiction.
For showing ($\Rightarrow$), let $\lambda \in \mathbb{R}$ such that $c := \inf_n |x(n)-\lambda| > 0$. Then $x(n) \neq \lambda$ for all $n = 1,2...$ This implies that $x-\lambda$ is injective. On the other hand, for any $\eta \in \ell_2$, if $(x(n)-\lambda)\xi(n) = \eta(n)$ for all n, then we have $\xi(n) = \frac{\eta(n)}{x(n)-\lambda}$ and so, $|\xi(n)| \leq \frac{|\eta(n)|}{c}$. This gives $\xi \in \ell_2$. Therefore, $x-\lambda$ is surjective and thus, $x-\lambda$ is invertible. Hence, $\lambda \notin \sigma(x)$.
From this, the last assertion follows because $\lambda \in \sigma(x)$ if and only if $\lambda = x_n$ for some n or there is a subsequence (x_{n_k}) of (x_n) that converges to λ.

Theorem 18.8. *Let $T \in B(H)$ be a selfadjoint operator. Put*

$$M(T) := \sup_{\|x\|=1} (Tx, x) \quad and \quad m(T) = \inf_{\|x\|=1} (Tx, x).$$

For convenience, we also write $M = M(T)$ and $m = m(T)$ if there is no confusion.
Then we have

(i) : $\|T\| = \max\{|m|, |M|\}$.
(ii) : $\{m, M\} \subseteq \sigma(T)$.
(iii) : $\sigma(T) \subseteq [m, M]$.

Proof. Note that m and M are well defined because (Tx, x) is real for all $x \in H$ by Proposition 16.13 (ii). In addition, Part(i) can be obtained by using Lemma 16.13 (ii) again.
For Part (ii), we first claim that $M \in \sigma(T)$ if $T \geq 0$. Note that $0 \leq m \leq M = \|T\|$ in this case by Lemma 16.13. Then there is a sequence (x_n) in H with $\|x_n\| = 1$ for all n such that $(Tx_n, x_n) \to M = \|T\|$. Then we have

$$\|(T - M)x_n\|^2 = \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M(Tx_n, x_n) \leq \|T\|^2 + M^2 - 2M(Tx_n, x_n) \to 0.$$

Hence, by Lemma 18.1 we have shown that $T - M$ is not invertible and hence $M \in \sigma(T)$ if $T \geq 0$.
Now for any selfadjoint operator T if we consider $T - m$, then $T - m \geq 0$. Thus we have $M - m = M(T - m) \in \sigma(T - m)$ by the previous case. Clearly, we have $\sigma(T - c) = \sigma(T) - c$ for all $c \in \mathbb{C}$. Therefore we have $M \in \sigma(T)$ for any self-adjoint operator.
We claim that that $m(T) \in \sigma(T)$. Note that $M(-T) = -m(T)$. Thus, we have $-m(T) \in \sigma(-T)$. It is clear that $\sigma(-T) = -\sigma(T)$. Then $m(T) \in \sigma(T)$.
Finally, we want to show $\sigma(T) \subseteq [m, M]$.
Indeed, since $T - m \geq 0$, then by Proposition 18.5, we have $\sigma(T) - m = \sigma(T - m) \subseteq [0, \infty)$. This gives $\sigma(T) \subseteq [m, \infty)$.
On the other hand, we consider $M - T \geq 0$. Then we get $M - \sigma(T) = \sigma(M - T) \subseteq [0, \infty)$. This implies that $\sigma(T) \subseteq (-\infty, M]$. The proof is complete. □

19. Appendix: $\sigma(T) \neq \emptyset$

Let X be a complex Banach space. In this appendix, we will show that the spectrum $\sigma(T)$ is non-empty for any $T \in B(X)$.
First we recall some basic result in Complex Analysis. Students can refer to any standard text book of Complex Analysis, see for example [1].
A function $g : \mathbb{C} \to \mathbb{C}$ is called an *entire function* if g is differentiable on $\mathbb{C}$, i.e., the following limit exists for all $c \in \mathbb{C}$

$$g'(c) := \lim_{z \to c} \frac{g(z) - g(c)}{z - c}.$$

The following result is one of important properties of entire functions (see [1, p.122]).

Theorem 19.1. Liouville's Theorem *Every bounded entire function is a constant function.*

Theorem 19.2. *Using the notion as before, let $T \in B(X)$. Then the spectrum $\sigma(T) \neq \emptyset$.*

Proof. Assume that $\sigma(T) = \emptyset$. Fix $f \in B(X)^*$, define the map $g(z) := f((z - T)^{-1})$ is defined for all $z \in \mathbb{C}$. Note that g is continuous on $\mathbb{C}$ by considering the composition $\lambda \in \mathbb{C} \mapsto \lambda - T \mapsto (\lambda - T)^{-1} \in B(X)$ and using Proposition 17.5 (iv). Moreover, we have $\lim_{z \to \infty} |g(z)| = 0$. Thus, g is a bounded function on $\mathbb{C}$. On the other hand, if we fix a point $c \in \mathbb{C}$, then we see that

$$\lim_{z \to c} \frac{g(z) - g(c)}{z - c} = -f((c - T)^{-1}).$$

Therefore, g is a bounded entire function. By the Liouville's Theorem, $f((z-T)^{-1})$ is a constant function on $\mathbb{C}$. Then the Hahn-Banach Theorem implies that the function $z \in \mathbb{C} \mapsto (z-T)^{-1} \in B(X)$ is constant on $\mathbb{C}$. It leads to a contradiction. □

20. Appendix: Existence of the square root of a positive operator

This section is based on the note of the course Functional Analysis taught by my teacher Dr. Chow Hing Lun in 1984-85 when I was an undergraduate student in the CUHK.
Throughout this section, let H be a complex Hilbert space and let T be a positive bounded operator on H. The aim of this section is to show that there is a unique positive operator S (called the *square root* of T) on H such that $S^2 = T$. The main feature of the proof here is without using the functional calculus.

Proposition 20.1. *Let $S, T \in B(H)$ such that $ST = TS$. If S, T both are positive operators, then so is ST.*

Proof. If $S = 0$, then the assertion is clear. Now we assume that $S \neq 0$. Put $S_1 := \frac{S}{\|S\|}$. Set

$$S_{n+1} := S_n - S_n^2$$

for $n = 1, 2, ...$
Claim 1: $0 \leq S_n \leq I$ for all $n = 1, ...$ The assertion will be obtained by induction on n. Notice that as $n = 1$, clearly we have $0 \leq S_1 \leq I$. Suppose that the Claim 1 is true for n, i.e., $0 \leq S_n \leq I$ and thus, we have $0 \leq I - S_n \leq I$. This implies that for all $x \in H$ we have $(S_n^2(I - S_n)x, x) = ((I - S_n)S_n x, S_n x) \geq 0$. This gives $S_n^2(I - S_n) \geq 0$. Similarly, we have $S_n(I - S_n)^2 \geq 0$. Hence, we have $0 \leq S_n^2(I - S_n) + S_n(I - S_n)^2 = S_n - S_n^2 = S_{n+1}$. On the other hand, we have $0 \leq (I - S_n) + S_n^2 = I - S_{n+1}$ because $S_n^2 \geq 0$ and $I - S_n \geq 0$. Therefore Claim 1 follows from the induction.
The proof will be complete if we show that $(STx, x) \geq 0$ for all $x \in H$.
In fact, notice that we have

$$S_1 = S_1^2 + S_2 = S_1^2 + S_2^2 + S_3 = \cdots = S_1^2 + \cdots + S_n^2 + S_{n+1}.$$

This implies that

$$S_1^2 + \cdots + S_n^2 = S_1 - S_{n+1} \leq S_1$$

for all $n = 1, 2..$ because $S_{n+1} \geq 0$. Thus, we have

$$\sum_{k=1}^{n} \|S_k x\|^2 = \sum_{k=1}^{n} (S_k x, S_k x) = \sum_{k=1}^{n} (S_k^2 x, x) \leq (S_1 x, x)$$

for all n. This gives $\sum_{k=1}^{\infty} \|S_k x\|^2 < \infty$ and so, $S_n x \to 0$. This implies that

$$(\sum_{k=1}^{n} S_k^2)x = S_1(x) - S_{n+1}(x) \to 0$$

for all $x \in H$ and so we have $\sum_{k=1}^{\infty} S_k^2(x) = S_1(x)$ for all $x \in H$. Finally, we complete the proof by the following

$$(STx, x) = \|S\|(TS_1 x, x) = \|S\| \sum_{k=1}^{\infty} (TS_k^2 x, x) = \|S\| \sum_{k=1}^{\infty} (TS_k x, S_k x) \geq 0$$

for all $x \in H$. □

Proposition 20.2. *Let T_n, $n = 1, 2, ...$ and K be the bounded linear selfadjoint operators on H. Suppose that*

(1) $T_1 \leq T_2 \leq \cdots \leq K$.

(2) $T_nT_m = T_nT_m$ and $KT_n = T_nK$ for all $m, n = 1, 2.....$

Then there is a bounded selfadjoint operator T on H with $T \leq K$ such that $\lim T_nx = Tx$ for all $x \in H$.

Proof. Now let $S_n := K - T_n$ for $n = 1, 2, ...$ Then $0 \leq S_n$ for all $n = 1, 2, ...$ By using Proposition 20.1, we see that $S_m^2 - S_nS_m = (S_m - S_n)S_m \geq 0$ and hence, $S_m^2 \geq S_nS_m$ for $n \geq m$. Similarly, we also have $S_nS_m \geq S_n^2$ for $n \geq m$. Therefore, we have

$$S_m^2 \geq S_nS_m \geq S_n^2 \tag{20.1}$$

for $n \geq m$. Thus, $((S_m^2x, x))_{m=1}^{\infty}$ is a decreasing sequence of non-negative numbers and so $\lim(S_n^2x, x)$ exists for all $x \in H$. Moreover since S_n and S_m commutes to each other, Eq 20.1 gives

$$\begin{aligned}
\|S_mx - S_nx\|^2 &= ((S_m - S_n)^2x, x) \\
&= (S_m^2x, x) - 2(S_mS_nx, x) + (S_m^2x, x) \\
&\leq (S_m^2x, x) - (S_n^2x, x) \to 0
\end{aligned}$$

for $n \geq m$ and for all $x \in H$. This implies that (S_nx) is a Cauchy sequence and hence, $\lim S_nx$ exists for all $x \in H$. This implies that $T(x) := \lim T_n(x) = K - \lim S_nx$ exists for all $x \in H$. The Uniform Boundedness Theorem tells us that $T \in B(H)$. In addition T is selfadjoint because each T_n is selfadjoint. The proof is complete. □

We now come to the main result in this section.

Theorem 20.3. *If T is a bounded positive operator on H, then there is a unique positive operator S such that $S^2 = T$. In this case, we call S the square root of T.*

Proof. We show the existence first.
Clearly, we may assume that $T \neq 0$ and $T \leq I$ by considering the operator $\frac{T}{\|T\|}$. Put $S_0 = 0$ and

$$S_n = S_{n-1} + \frac{1}{2}(T - S_{n-1}^2)$$

for $n = 1, 2, ...$ Then S_n is a polynomial of T and so, all S_n's are selfadjoint operators and commute to each other. Notice that since $0 < T \leq I$ and by the definition of S_n, we have

$$I - S_n = I - S_{n-1} - \frac{1}{2}(T - S_{n-1}^2) = \frac{1}{2}(I - S_{n-1})^2 + \frac{1}{2}(I - T) \geq 0.$$

Thus $S_n \leq I$ for all $n = 0, 1, 2....$ On the other hand, we have

$$S_{n+1} - S_n = S_n + \frac{1}{2}(T - S_n^2) - S_{n-1} - \frac{1}{2}(T - S_{n-1}^2) = (S_n - S_{n-1})(I - \frac{1}{2}(S_n + S_{n-1})) \tag{20.2}$$

for all $n = 0, 1, 2....$ Since $S_n \leq I$, $I - \frac{1}{2}(S_n + S_{n-1}) \geq 0$. Using Proposition 20.1 and the Eq 20.2, we can apply induction on n to see that $0 = S_0 \leq \cdots \leq S_n \leq S_{n+1} \leq \cdots \leq I$ for all $n = 0, 1, 2....$ Proposition 20.2 tells us that $Sx := \lim S_nx$ exists for all $x \in H$ and $S \in B(H)$. In addition S is positive because $S_n \geq 0$ for all $n = 0, 1, 2....$ Also, since $S_nx = S_{n-1}x + \frac{1}{2}(T - S_{n-1}^2)x$ for all $x \in H$, by taking $n \to \infty$, we see that $Tx = S^2x$ for all x. Thus the operator S is as desired.
Finally, we show the uniqueness.
Now let R be another positive bounded operator on H such that $R^2 = T$. Notice that $RT = R^3 = TR$. This implies that $RS = SR$ because S is the $\| \cdot \|$-limit of the polynomials of T by the above construction of S. Now we take any $x \in H$ and put $y := (S - R)x$. Then we have

$$0 \leq (Sy, y) + (Ry, y) = ((S + R)(S - R)x, y) = ((S^2 - R^2)x, y) = 0.$$

This implies that $(Sy, y) = (Ry, y) = 0$ because both are non-negative numbers. On the other hand, since $S \leq 0$, by above there is another positive operator W such that $W^2 = S$, and so we have $0 = (Sy, y) = (Wy, Wy)$ that gives $Sy = 0$. Similarly, we also have $Ry = 0$. Finally, we have

$$\|(S - R)x\|^2 = ((S - R)^2 x, x) = ((S - R)y, x) = 0.$$

Thus, $S = R$ as desired. The proof is complete. □

21. Compact operators on a Hilbert space

Throughout this section, let H be a complex Hilbert space.

Definition 21.1. *A linear operator $T : H \to H$ is said to be compact if for every bounded sequence (x_n) in H, $(T(x_n))$ has a norm convergent subsequence.*
Write $K(H)$ for the set of all compact operators on H and $K(H)_{sa}$ for the set of all compact selfadjoint operators.

Remark 21.2. *Let U be the closed unit ball of H. Clearly, T is compact if and only if the norm closure $\overline{T(U)}$ is a compact subset of H. Thus if T is compact, then T is bounded automatically because every compact set is bounded. In particular, if T has finite rank, that is* $\dim\ imT < \infty$, *then T must be compact because every closed and bounded subset of a finite dimensional normed space is compact. In addition, clearly we have the following result.*

Proposition 21.3. *The identity operator $I : H \to H$ is compact if and only if* $\dim H < \infty$.

Example 21.4. *Let $H = \ell^2(\{1, 2...\})$. Define $Tx(k) := \frac{x(k)}{k}$ for $k = 1, 2....$ Then T is compact. In fact, if we let (x_n) be a bounded sequence in ℓ^2, then by the diagonal argument, we can find a subsequence $y_m := Tx_m$ of Tx_n such that $\lim\limits_{m\to\infty} y_m(k) = y(k)$ exists for all $k = 1, 2...$ Let $L := \sup_n \|x_n\|_2^2$. Since $|y_m(k)|^2 \leq \frac{L}{k^2}$ for all m, k, we have $y \in \ell^2$. Now let $\varepsilon > 0$. Then one can find a positive integer N such that $\sum_{k\geq N} 4L/k^2 < \varepsilon$. Thus we have*

$$\sum_{k\geq N} |y_m(k) - y(k)|^2 < \sum_{k\geq N} \frac{4L}{k^2} < \varepsilon$$

for all m. On the other hand, since $\lim\limits_{m\to\infty} y_m(k) = y(k)$ for all k, we can choose a positive integer M such that

$$\sum_{k=1}^{N-1} |y_m(k) - y(k)|^2 < \varepsilon$$

for all $m \geq M$. Finally, we have $\|y_m - y\|_2^2 < 2\varepsilon$ for all $m \geq M$.

Theorem 21.5. *Let $T \in B(H)$. Then T is compact if and only if T maps every weakly convergent sequence in H to a norm convergent sequence.*

Proof. We first assume that $T \in K(H)$. Let (x_n) be a weakly convergent sequence in H. Since H is reflexive, (x_n) is bounded by the Uniform Boundedness Theorem. Thus we can find a subsequence (x_j) of (x_n) such that (Tx_j) is norm convergent. Let $y := \lim_j Tx_j$. We claim that $y = \lim_n Tx_n$. Suppose that $y \neq \lim_n Tx_n$. Then by the compactness of T again, we can find a subsequence (x_i) of (x_n) such that Tx_i converges to y' with $y \neq y'$. Thus there is $z \in H$ such that $(y, z) \neq (y', z)$. On the other hand, if we let x be the weakly limit of (x_n), then $(x_n, w) \to (x, w)$ for all $w \in H$. Thus we have

$$(y, z) = \lim_j (Tx_j, z) = \lim_j (x_j, T^*(z)) = (x, T^*z) = (Tx, z).$$

Similarly, we also have $(y', z) = (Tx, z)$ and hence $(y, z) = (y', z)$ that contradicts to the choice of z.
For the converse, let (x_n) be a bounded sequence. Then by Theorem 15.12, (x_n) has a weakly convergent subsequence. Thus $T(x_n)$ has a norm convergent subsequence by the assumption. Thus T is compact. □

Proposition 21.6. *Let $S, T \in K(H)$. Then we have*
(i) : $\alpha S + \beta T \in K(H)$ for all $\alpha, \beta \in \mathbb{C}$;
(ii) : TQ and $QT \in K(H)$ for all Q in $B(H)$;
(iii) : $T^* \in K(H)$.
Moreover $K(H)$ is normed closed in $B(H)$, and hence $K(H)$ is a closed $*$-ideal of $B(H)$.

Proof. (i) and (ii) are clear.
For property (iii), let (x_n) be a bounded sequence. Then (T^*x_n) is also bounded. Thus TT^*x_n has a convergent subsequence $TT^*x_{n_k}$ by the compactness of T. Note that we have

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = (TT^*(x_{n_k} - x_{n_l}), x_{n_k} - x_{n_l})$$

for all n_k, n_l. This implies that $(T^*x_{n_k})$ is a Cauchy sequence and thus is convergent.
Finally we want to show that $K(H)$ is closed. Let (T_m) be a sequence in $K(H)$ such that $T_m \to T$ in norm. Let (x_n) be a bounded sequence in H. Then by the diagonal argument there is a subsequence (x_{n_k}) of (x_n) such that $\lim_k T_m x_{n_k}$ exists for all m. Now let $\varepsilon > 0$. Since $\lim_m T_m = T$, there is a positive integer N such that $\|T - T_N\| < \varepsilon$. On the other hand, there is a positive integer K such that $\|T_N x_{n_k} - T_N x_{n_{k'}}\| < \varepsilon$ for all $k, k' \geq K$. Thus we can now have

$$\|Tx_{n_k} - Tx_{n_{k'}}\| \leq \|Tx_{n_k} - T_N x_{n_k}\| + \|T_N x_{n_k} - T_N x_{n_{k'}}\| + \|T_N x_{n_{k'}} - Tx_{n_{k'}}\| \leq (2L+1)\varepsilon$$

for all $k, k' \geq K$ where $L := \sup_n \|x_n\|$. Thus $\lim_k Tx_{n_k}$ exists. We can now conclude that $T \in K(H)$. □

Example 21.7. *Let $k(z, w) \in C(\mathbb{T} \times \mathbb{T})$. Define an operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by*

$$T\xi(z) := \int_{\mathbb{T}} k(z, w)\xi(w)dw$$

for $z \in \mathbb{T}$ and $\xi \in L^2(\mathbb{T})$. Then T is a compact operator.

Proof. Clearly, we have $\|T\| \leq \|k\|_\infty$. On the other hand, Stone-Weiestrass Theorem tells us the polynomials of $(z, \bar{z}; w, \bar{w})$ are $\|\cdot\|_\infty$-dense in $C(\mathbb{T} \times \mathbb{T})$. Therefore, by using Proposition 21.6, it suffices to show for the case $k(z, w) = \sum_{i,j=1}^N a_{ij}(z, \bar{z}) w^i \bar{w}^j$ where $a_{ij}(z, \bar{z})$ is a polynomial of $(z, \bar{z})$ of degree N. From this, we have

$$T\xi(z) = \sum_{i,j=1}^N a_{ij}(z, \bar{z}) \int_{\mathbb{T}} w^i \bar{w}^j \xi(w)dw$$

for $\xi \in L^2(\mathbb{T})$. Thus, $T(\xi) \in span\{z^i \bar{z}^j : 0 \leq i, j \leq N\}$ which is of finite dimension for all $\xi \in L^2(\mathbb{T})$. This implies that T has finite dimensional range and thus, T is compact. The proof is complete. □

Corollary 21.8. *Let $T \in K(H)$. If $\dim H = \infty$, then $0 \in \sigma(T)$.*

Proof. Suppose that $0 \notin \sigma(T)$. Then T^{-1} exists in $B(H)$. Proposition 21.6 gives $I = TT^{-1} \in K(H)$. This implies $\dim H < \infty$. □

Proposition 21.9. *Let $T \in K(H)$ and let $c \in \mathbb{C}$ with $c \neq 0$. Then $T - c$ has a closed range.*

Proof. Note that $\frac{1}{c}T \in K(H)$. Thus if we consider $\frac{1}{c}T-I$, we may assume that $c = 1$. Let $S = T-I$. Let (x_n) be a sequence in H such that $Sx_n \to x \in H$ in norm. By considering the orthogonal decomposition $H = \ker S \oplus (\ker S)^\perp$, we write $x_n = y_n \oplus z_n$ for $y_n \in \ker S$ and $z_n \in (\ker S)^\perp$. We first claim that (z_n) is bounded. Suppose that (z_n) is unbounded. By considering a subsequence of (z_n), we may assume that we may assume that $\|z_n\| \to \infty$. Put $v_n := \frac{z_n}{\|z_n\|} \in (\ker S)^\perp$. Since $Sz_n = Sx_n \to x$, we have $Sv_n \to 0$. On the other hand, since T is compact, and (v_n) is bounded, by passing a subsequence of (v_n), we may also assume that $Tv_n \to w$. Since $S = T - I$, $v_n = Tv_n - Sv_n \to w - 0 = w \in (\ker S)^\perp$. In addition from this we have $Sv_n \to Sw$. On the other hand, we have $Sw = \lim_n Sv_n = \lim_n Tv_n - \lim_n v_n = w - w = 0$. Thus $w \in \ker S \cap (\ker S)^\perp$. It follows that $w = 0$. However, since $v_n \to w$ and $\|v_n\| = 1$ for all n. It leads to a contradiction. Thus (z_n) is bounded.

Finally we are going to show that $x \in imS$. Now since (z_n) is bounded, (Tz_n) has a convergent subsequence (Tz_{n_k}). Let $\lim_k Tz_{n_k} = z$. Then we have

$$z_{n_k} = Sz_{n_k} - Tz_{n_k} = Sx_{n_k} - Tz_{n_k} \to x - z.$$

It follows that $x = \lim_k Sx_{n_k} = \lim_k Sz_{n_k} = S(x - z) \in imS$. The proof is complete. □

Theorem 21.10. Fredholm Alternative Theorem : *Let $T \in K(H)_{sa}$ and let $0 \neq \lambda \in \mathbb{C}$. Then $T - \lambda$ is injective if and only if $T - \lambda$ is surjective.*

Proof. Since T is selfadjoint, $\sigma(T) \subseteq \mathbb{R}$. Thus if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $T - \lambda$ is invertible. Thus the result holds automatically.

Now consider the case $\lambda \in \mathbb{R} \setminus \{0\}$.

Then $T - \lambda$ is also selfadjoint. From this and Proposition 16.14, we have $\ker(T-\lambda) = (im(T-\lambda))^\perp$ and $(\ker(T - \lambda))^\perp = \overline{im(T - \lambda)}$.

Thus the proof is complete immediately by using Proposition 21.9. □

Corollary 21.11. *Let $T \in K(H)_{sa}$. Then we have $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. Consequently if the values $m(T)$ and $M(T)$ which are defined in Theorem 18.8 are non-zero, then both are the eigenvalues of T and $\|T\| = \max_{\lambda \in \sigma_p(T)} |\lambda|$.*

Proof. It follows immediately from the Fredholm Alternative Theorem. This, together with Theorem 18.8, implies the last assertion. □

Example 21.12. *Let $T \in B(\ell^2)$ be defined as in Example 21.4. We have shown that $T \in K(\ell^2)$ and T is selfadjoint. Then by Corollary 21.11 and Corollary 21.8, we see that $\sigma(T) = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$.*

Lemma 21.13. *Let $T \in K(H)_{sa}$ and let $E_\lambda := \{x \in H : Tx = \lambda x\}$ for $\lambda \in \sigma(T) \setminus \{0\}$, that is the eigenspace of T corresponding to λ. Then $\dim E_\lambda < \infty$.*

Proof. It is because the restriction $T|E_\lambda : E_\lambda \to E_\lambda$ is also a compact operator on E_λ, then $\dim E_\lambda < \infty$ for all $\lambda \in \sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. □

Theorem 21.14. *Let $T \in K(H)_{sa}$. Suppose that $\dim H = \infty$. Then $\sigma(T) = \{\lambda_k : k = 1, ..., N\} \cup \{0\}$, where $1 \leq N \leq \infty$ and (λ_n) is a sequence of non-zero real numbers with $|\lambda_1| \geq |\lambda_2| \geq \cdots$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Moreover, if (λ_n) is an infinite sequence, then $|\lambda_n| \downarrow 0$.*

Proof. Note that since $\dim H = \infty$, $0 \in \sigma(T)$. In addition we have $\|T\| = \max(|M(T)|, |m(T)|)$ and $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. Thus by Corollary 21.11, there is $|\lambda_1| = \max\limits_{\lambda \in \sigma_p(T)} |\lambda| = \|T\|$. Since $\dim E_{\lambda_1} < \infty$, then $E_{\lambda_1}^{\perp} \neq 0$. By considering the restriction of $T_2 := T|E_{\lambda_1}^{\perp}$, if $T_2 \neq 0$, then there is $0 \neq |\lambda_2| = \max_{\lambda \in \sigma_p(T_2)} |\lambda| = \|T_2\|$. Note that $\lambda_2 \in \sigma_p(T)$ and $|\lambda_2| \leq |\lambda_1|$ because $\|T_2\| \leq \|T\|$. To repeat the same step, if $T_{N+1} = 0$ for some N, then $0 \in \sigma_p(T)$. Otherwise, we can get an infinite sequence (λ_n) such that $(|\lambda_n|)$ is decreasing.
Now we claim that if (λ_n) is an infinite sequence, then $\lim_n |\lambda_n| = 0$.
Otherwise, there is $\eta > 0$ such that $|\lambda_n| \geq \eta$ for all n. If we let $v_n \in E_{\lambda_n}$ with $\|v_n\| = 1$ for all n. Note that since $\dim H = \infty$ and $\dim E_\lambda < \infty$, for any $\lambda \in \sigma_p(T) \setminus \{0\}$, there are infinite many λ_n's. Then $w_n := \frac{1}{|\lambda_n|} v_n$ is a bounded sequence and $\|Tw_n - Tw_m\|^2 = \|v_n - v_m\|^2 = 2$ for $m \neq n$. This is a contradiction since T is compact. Thus $\lim_n |\lambda_n| = 0$.
Finally we need to check $\sigma(T) = \{\lambda_1, \lambda_2, ...\} \cup \{0\}$.
In fact, let $\mu \in \sigma_p(T)$. Since $|\lambda_1| = \|T\| \geq |\mu|$, $|\lambda_{m+1}| < |\mu| \leq |\lambda_m|$. Note that we have $E_\alpha \perp E_\beta$ for α and β in $\sigma_p(T)$ with $\alpha \neq \beta$. Then by the construction of λ_n's, we have $\mu = \lambda_m$. For example, if $|\lambda_2| < |\mu| \leq |\lambda_1|$ and $\mu \neq \lambda_1$, then $E_\mu \perp E_{\lambda_1}$. Hence, we have $E_\mu \subseteq (E_{\lambda_1})^{\perp}$. Then by the construction of λ_2, that is $|\lambda_2| = \|T_2\| \geq |\mu|$ which leads to a contradiction. Thus, if $|\lambda_2| < |\mu| \leq |\lambda_1|$, then $\mu = \lambda_1$. The proof is complete. ☐

Theorem 21.15. Spectral Decomposition Theorem: *Let $T \in K(H)_{sa}$ and let $(\lambda_n)_{n=1}^N$, ($1 \leq N \leq \infty$), be a sequence of given as in Theorem 21.14. For each $\lambda \in \sigma_p(T) \setminus \{0\}$, put $d(\lambda) := \dim E_\lambda < \infty$. Let $\{e_{\lambda,i} : i = 1, ..., d(\lambda)\}$ be an orthonormal basis for E_λ. Then we have the following orthogonal decomposition:*

$$H = \ker T \oplus \bigoplus_{n=1}^{N} E_{\lambda_n}. \tag{21.1}$$

Moreover $\mathcal{B} := \{e_{\lambda,i} : \lambda \in \sigma_p(T) \setminus \{0\}; i = 1, .., d(\lambda)\}$ forms an orthonormal basis of $\overline{T(H)}$, and we have

$$Tx = \sum_{n=1}^{N} \sum_{i=1}^{d(\lambda_n)} \lambda_n (x, e_{\lambda_n,i}) e_{\lambda_n,i} \tag{21.2}$$

for all $x \in H$.
In addition, if $N = \infty$, then the series $\sum\limits_{n=1}^{\infty} \lambda_n P_n$ norm converges to T, where P_n is the orthogonal projection from H onto E_{λ_n}, that is, $P_n(x) := \sum\limits_{i=1}^{d(\lambda_n)} (x, e_{\lambda_n,i}) e_{\lambda_n,i}$, for $x \in H$.

Proof. Put $E = \bigoplus_{n=1}^N E_{\lambda_n}$. Clearly, we have $\ker T \subseteq E^{\perp}$. On the other hand, if the restriction $T_0 := T|E^{\perp} \neq 0$, then there exists an non-zero element $\mu \in \sigma_p(T_0) \subseteq \sigma_p(T)$ because $T_0 \in K(E^{\perp})$. It is absurd because $\mu \neq \lambda_i$ for all i. Thus $T|E^{\perp} = 0$ and hence $E^{\perp} \subseteq \ker T$. Therefore, we have the decomposition (21.1). Moreover, from this we see that the family $\mathcal{B}$ forms an orthonormal basis of $(\ker T)^{\perp}$. On the other hand, we have $(\ker T)^{\perp} = \overline{imT^*} = \overline{imT}$. Therefore, $\mathcal{B}$ is an orthonormal basis for $\overline{T(H)}$ and Equation 21.2 follows.
For the last assertion, it needs to show that the series $\sum_{n=1}^{\infty} \lambda_n P_n$ converges to T in norm. Note that if we put $S_m := \sum_{n=1}^{m} \lambda_n P_n$, then by the decomposition (21.1), $\lim\limits_{m \to \infty} S_m x = Tx$ for all $x \in H$. Thus it suffices to show that $(S_m)_{m=1}^{\infty}$ is a Cauchy sequence in $B(H)$. In fact we have

$$\|\lambda_{m+1} P_{m+1} + \cdots\cdots + \lambda_{m+p} P_{m+p}\| = |\lambda_{m+1}|$$

for all $m, p \in \mathbb{N}$ because $E_{\lambda_n} \perp E_{\lambda_m}$ for $m \neq n$ and $|\lambda_n|$ is decreasing. This gives that (S_n) is a Cauchy sequence since $|\lambda_n| \downarrow 0$ as $N = \infty$. The proof is complete. □

Corollary 21.16. *$T \in K(H)$ if and only if T can be approximated by finite rank operators.*

Proof. The sufficient condition follows immediately from Proposition 21.6.
Conversely, for a general compact operator T, we can consider the decomposition:

$$T = \frac{1}{2}(T + T^*) + i(\frac{1}{2i}(T - T^*)).$$

Note that $Re(T) := \frac{1}{2}(T+T^*)$ (*call the real part of* T) and $Im(T) := \frac{1}{2i}(T-T^*)$ (*call the imaginary part of* T) both are the self-adjoint compact operators. From this, we see that the T can be approximated by finite ranks operators by using Theorem 21.15. □

22. Unbounded operators

Throughout this section, let H be a complex Hilbert space. An operator T on H means that T is a linear operator defined in a vector subspace of T (it is not necessarily bounded). We write $D(T)$ for the domain of T. We say that T is *densely defined* if the domain $D(T)$ is dense in H. An operators S is said to be an extension of T if $D(T) \subseteq D(S)$ and $Sx = Tx$ for all $x \in D(T)$, denoted it by $T \subset S$.
In addition, if T and S are operators on H, then we naturally define the domains of the following operations.

(i) $D(T + S) := D(T) \cap D(S)$.
(ii) $D(S \circ T) := \{x \in D(T) : Tx \in D(S)\}$.

Definition 22.1. *Let T be a densely defined operator on H. Put*

$$D(T^*) := \{y \in H : \text{ there is } z \in H \text{ such that } (Tx, y) = (x, z) \text{ for all } x \in D(T)\}.$$

Clearly, $D(T^*)$ is a vector subspace of H. In addition, since T is densely defined, for each element $y \in D(T^*)$, there is a unique element in H, denoted it by T^*y, satisfying

$$(Tx, y) = (x, T^*y)$$

for all $x \in D(T)$. We call T^* the adjoint operator of T.
We call an operator T symmetric (resp. self-adjoint) if $T \subset T^*$ (resp. $T = T^*$).
Note that T is symmetric if and only if we have

$$(Tx, y) = (x, Ty)$$

for all $x, y \in D(T)$.

Proposition 22.2. *Let S, T be the operators on H. Assume that T, S and ST are densely defined. Then $T^*S^* \subset (ST)^*$.*

Proof. We first claim that $T^*S^* \subset (ST)^*$. Let $x \in D(ST)$ and $y \in D(T^*S^*)$. Then S^*y is defined and $S^*y \in D(T^*)$. Since $x \in D(ST)$ we have $x \in D(T)$ and $Tx \in D(S)$. Thus we have

$$(STx, y) = (Tx, S^*y) = (x, T^*S^*y).$$

This implies that $y \in D(ST)^*)$ and $(ST)^*(y) = T^*S^*y$ and hence $T^*S^* \subset (ST)^*$. □

Example 22.3. *First we recall that a function $f : [a,b] \to \mathbb{C}$ is called an indefinite integral if there is an element $\varphi \in L^1[a,b]$ such that*

$$f(x) = f(a) + \int_a^x \varphi(t)dt$$

for all $x \in [a,b]$, where dt is the Lebesgue measure on $[a,b]$. In this case we have $f'(x) = \varphi(x)$ almost everywhere in (a,b).
Let

$$D := \{f : [a,b] \to \mathbb{C} : \ f \text{ is an indefinite integral with } f(a) = f(b) \text{ and } f' \in L^2[a,b]\}.$$

Note that D is dense subspace of $L^2[a,b]$. Define an operator T with $D(T) = D$ by

$$Tf := if'.$$

for $f \in D$. We claim that T is self-adjoint. The proof is divided by several steps.
Claim 1*: $T \subset T^*$.*
In fact, let $f, g \in D$. Then we have

$$\begin{aligned}(Tf,g) &= \int_a^b if'(t)\overline{g(t)}dt \\ &= \int_a^b i\overline{g(t)}df(t) \\ &= if(t)\overline{g(t)}|_a^b - i\int_a^b f(t)\overline{g'(t)}dt \\ &= \int_a^b f(t)\overline{ig'(t)}dt = (f,Tg).\end{aligned} \tag{22.1}$$

Therefore, the Claim 1 follows. Next we want to show $D(T^*) \subseteq D(T)$.
Let $g \in D(T^*)$. Put $\varphi := T^*g \in L^2[a,b]$. Note that $\varphi \in L^1[a,b]$ because $L^2[a,b] \subseteq L^1[a,b]$. Thus, $\Phi(x) := \int_a^x \varphi(t)dt$ for $x \in [a,b]$ is an indefinite integral of φ.
Claim 2*: There is a constant c so that $g(t) + i\Phi(t) = c$ for all $t \in [a,b]$. Note that for any $f \in D$, we have*

$$\begin{aligned}(Tf,g) &= (f,T^*g) \\ &= \int_a^b f(t)\overline{\varphi(t)}dt \\ &= \int_a^b f(t)d\overline{\Phi(t)} \\ &= f(b)\overline{\Phi(b)} - \int_a^b \overline{\Phi(t)}f'(t)dt \\ &= \overline{\Phi(b)} - (Tf,i\Phi).\end{aligned}$$

From this if we take $f \equiv 1 \in D$ in above, then $\Phi(b) = 0$. Therefore, we have

$$(Tf,g) = -(Tf,i\Phi)$$

for all $f \in D$. This implies that $(g + i\Phi)\perp im(T)$. If we let $\mathbf{1} \in L^2[a,b]$ be the function of constant one in $[a,b]$, then we have

$$(Tf,\mathbf{1}) = \int_a^b if'(t)dt = i(f(b) - f(a)) = 0$$

for all $f \in D$, hence $\mathbb{C}\mathbf{1}\perp im(T)$. On the other hand, note that for any $\xi \in L^2[a,b]$ if we put $\xi_1 = \xi - \int_a^b \xi(t)dt \in L^2[a,b]$, then $\int_a^b \xi_1(t)dt = 0$. Let $h(x) := i\int_a^x \xi_1(t)dt$. Then $h \in D$ and $Th = \xi_1$. Therefore, we have $L^2[a,b] = \mathbb{C}\mathbf{1} + im(T)$ and hence we have the orthogonal decomposition

$L^2[a,b] = \mathbb{C}\mathbf{1} \oplus \overline{im(T)}$. In particular, $(im(T))^\perp = \mathbb{C}\mathbf{1}$. This implies that $g + i\Phi = c$ for some constant c. Then $g' = -i\Phi' = -i\varphi \in L^2[a,b]$, so g is an indefinite integral because $g' \in L^1[a,b]$. Moreover, we see that $g(b) = g(a) = c$ because $\Phi(b) = \Phi(a) = 0$. We can now conclude that $g \in D$. The proof is complete.

Example 22.4. *Using the notation as in Example 22.3, we let*

$$D_1 := \{f \in D : f(a) = f(b) = 0\}.$$

Then D_1 is dense subspace of $L^2[a,b]$. Define $T_1 : D_1 \to L^2[a,b]$ by

$$T_1 f = if'$$

for $f \in D_1$. Then T_1 is symmetric but it is not self-adjoint.
By using the similar calculation as in Eq 22.1 in Example 22.3 above, we see that $T_1 \subset T_1^*$. Let $D_2 := \{f : [a,b] \to \mathbb{C} :\ f$ is an indefinite integral and $f' \in L^2[a,b]\ \}$. Then $D_2 \subseteq D(T_1^*)$. In fact, let $f \in D_1$ and $g \in D_2$, using the same argument as in Eq 22.1 again, we have

$$(T_1 f, g) = if(t)\overline{g(t)}|_a^b - i\int_a^b f(t)\overline{g'(t)}dt = \int_a^b f(t)\overline{ig'(t)}dt = (f, T_2 g)$$

because $f(a) = f(b) = 0$, where $T_2(g) := ig'$ for $g \in D_2$. Therefore $D(T_1) \subsetneq D(T_1^*)$ since $D(T_1) = D_1 \subsetneq D_2$. The proof is complete.

Definition 22.5. *An operator T on H is said to be closed if its graph of T, denoted it by $G(T) := \{(x, Tx) \in H \times H : x \in D(T)\}$, is closed in $H \times H$. More precisely, if (x_n) is a sequence in $D(T)$ such that $x_n \to x$ and $Tx_n \to y$, then $x \in D(T)$ and $Tx = y$.*

Define an operator $V : H \times H \to H \times H$ by $V(x,y) = (-y, x)$ for $(x,y) \in H \times H$. Then $(V(x,y), V(x',y')) = ((x,y),(x',y'))$ for all (x,y) and (x',y') in $H \times H$ and hence, the operator preserves the orthogonality on $H \times H$.

Proposition 22.6. *Using the notation as above, let T be a densely operator on H. Then $G(T^*) = (V(G(T)))^\perp$. Consequently, the adjoint operator T^* is closed. In particular, if T is self-adjoint, then T is closed.*

Proof. Note that for $x \in D(T^*)$ and $y \in D(T)$, we have $((x, T^*x), V(y, Ty)) = 0$ Therefore, we have $G(T^*) \subseteq (V(G(T)))^\perp$. On the other hand, if $(u,v)\perp(-Ty, y)$ for all $y \in D(T)$. Then we have $(v,y) = (u, Ty)$ and hence, $u \in D(T^*)$ and $T^*u = v$. Therefore, $(u,v) \in G(T^*)$. The proof is complete. ☐

Proposition 22.7. *Let T be a symmetric operator on H. Then the following statements are equivalent.*

(i) T is self-adjoint.
(ii) T is closed and $\ker(T^* \pm i) = \{0\}$.
(iii) $im(T \pm i) = H$.

Proof. For $(i) \Rightarrow (ii)$, assume that T is self-adjoint. Then by Proposition 22.6, T is closed. Next we show $\ker(T^* - i) = \{0\}$. Let $y \in D(T^*)$ such that $T^*y = iy$. Since $D(T) = D(T^*)$, we have $i(y,y) = (Ty, y) = (y, T^*y) = -i(y,y)$. Thus, $y = 0$. Similarly, we have $\ker(T^* + i) = \{0\}$.
For $(ii) \Rightarrow (iii)$, we first claim that $im(T+i)$ is dense in H. Let $z \perp im(T+i)$. Then $z\perp(T+i)x$ for all $x \in D(T)$, and thus we have $(Tx, z) = (x, -iz)$. This implies that $z \in D(T^*)$ and $T^*z = -iz$. Thus, $z \in \ker(T^* + i)$, so $z = 0$. Therefore, it suffices to show that $im(T - i)$ is closed. Let (x_n) be a sequence in $D(T)$ such that $\lim(T - i)x_n = y$. Since T is symmetric, we have

$$\|T(x_m - x_n) - i(x_m - x_n)\|^2 = \|T(x_m - x_n)\|^2 + \|(x_m - x_n)\|^2$$

for all m, n. From this we see that $u := \lim x_n$ and $v := \lim Tx_n$ both exist. T is closed by the assumption, so $u \in D(T)$ and $Tu = v$. Therefore, we have

$$y = \lim(Tx_n - ix_n) = v - iu = (T - i)u \in im(T - i).$$

Hence $im(T - i) = H$. Similarly, we have $im(T + i) = H$.
For the last implication $(iii) \Rightarrow (i)$, since $T \subset T^*$, we need to show that $D(T^*) \subseteq D(T)$. Let $u \in D(T^*)$. Since $im(T - i) = H$, there is an element $v \in D(T)$ such that

$$(T - i)v = (T^* - i)u.$$

Since $T \subset T^*$, we have $(T - i)v = (T^* - i)v$, thus, $v - u \in \ker(T^* - i)$. Then for any $z \in D(T)$, we have

$$((T + i)z, v - u) = (z, (T + i)^*(v - u)) = (z, (T^* - i)(v - u)) = 0.$$

$im(T + i) = H$ by assumption, so $u = v \in D(T)$. The proof is complete. □

Proposition 22.8. *Let T be a symmetric operator on H. Then there is the smallest closed extension of T, denoted it by $\overline{T}$. We call $\overline{T}$ the closure of T. In addition, $G(\overline{T}) = \overline{G(T)}$ and $\overline{T} = T^{**}$.*

Proof. Let $D(\overline{T}) := \{x \in H : (x, y) \in \overline{G(T)}$ for some $y \in H\}$. We first note for each element $x \in D(\overline{T})$, there is a unique element $y \in H$ so that $(x, y) \in \overline{G(T)}$. In fact, if $(x, y) \in \overline{G(T)}$, there is a sequence (x_n) in $D(T)$ such that $\lim x_n = x$ and $\lim Tx_n = y$. Note that for any $u \in D(T)$, since T is symmetric, we have

$$(Tu, x) = \lim(Tu, x_n) = \lim(u, Tx_n) = (u, y).$$

Therefore, y is uniquely determined by x because $D(T)$ is dense in H. Hence, we can define $\overline{T}x = y$ for $x \in D(\overline{T})$. Clearly, we have $G(\overline{T}) = \overline{G(T)}$ by the construction of $\overline{T}$, and hence $\overline{T}$ is closed. Moreover, we can directly show that $\overline{T}$ is the smallest closed extension of T.
For the last assertion, since $T \subset T^*$, T^* is densely defined, so $T^{**} := (T^*)^*$ is defined. Since $V^2 = -I$ and V is an isometry and an orthogonal preserver, by using Proposition 22.6, we have

$$\begin{aligned} G(T^{**}) &= [VG(T^*)]^{\perp} \\ &= V[G(T^*)^{\perp}] \\ &= V[\overline{V(G(T))}] \\ &= V^2\overline{(G(T))} \\ &= G(\overline{T}). \end{aligned}$$

Thus, $\overline{T} = T^{**}$. □

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